Chapter 16

Recursive Functions

16.4 References

16.1 Recursive Functions

16.1.1 Iterative versus Recursive

Definitions of the Factorial Function

```
factorial(n)  := n!

              := n * (n-1) * … * 2 * 1
```

$$:= \prod_{i=1}^{n} i$$

Iterative Implementation

$$\texttt{factorial(n)} := \prod_{i=1}^{n} i$$

Here is a typical iterative C++ implementation of the factorial function:

```
int factorial_iter (int n)
{
  int i, f=1;

  for(i=1; i <= n; i++) f *= i;

  return f;
}
```

Towards a Recursive Definition

$$\texttt{factorial(n) :=} \prod_{i=1}^{n} i \texttt{ = n * (n-1) * … * 2 * 1}$$

$$\texttt{factorial(n-1) :=} \prod_{i=1}^{n-1} i \texttt{ = (n-1) * (n-2) * … * 2 *1}$$

$$\texttt{factorial(n-2) :=} \prod_{i=1}^{n-2} i \texttt{ = (n-2) * … * 2 * 1}$$

…

$$\texttt{factorial(2) :=} \prod_{i=1}^{2} i \texttt{ = 2 * 1}$$

A Recursive Definition

$$factorial(n) := \begin{cases} 1 & \text{if } n = 1 \\ n \cdot factorial(n-1) & \text{if } n > 1 \end{cases}$$

Recursive Implementation of the Factorial Function (version 1)

```
int factorial_rec (int n)
{
  if(n == 1) return 1;

  return n * factorial_rec(n-1);
}
```

16.1.2 Comparing Iterative and Recursive Processes

Iterative Version

```
int factorial_iter (int n)
{
   int i, f=1;
   for(i=1; i <= n; i++) f *= i;
   return f;
}
```

Executing this program generates a linear iterative process:

Iteration	i	f	n
1	1	1	4
2	2	2	4
3	3	6	4
4	4	24	4
5	5		

Recursive Version

```
int factorial_rec(int n)
{
  (n == 1) ? 1 : (n * factorial_rec(n-1));
}
```

This generates a linear recursive process, as the following example shows:

```
fac_rec(5)
(5 * fac_rec(4))
(5 * (4 * fac_rec(3)))
(5 * (4 * (3 * fac_rec(2))))
(5 * (4 * (3 * (2 * fac_rec(1)))))
(5 * (4 * (3 * (2 * 1))))
(5 * (4 * (3 * 2)))
(5 * (4 * 6))
(5 * 24)
(120)
```

16.2 Further Examples with Recursion

16.2.1 String Reversion

The following example uses a recursive function to print a string backwards.

```
void reverse(char *s)
{
  if(*s)
    reverse(s+1)
  else
    return;
}

void main()
{
  cout << reverse("Left to right");
}
```

16.2.2 Recursion over Arrays

A function for adding elements *m* through *n* of an array, can be defined as follows:

- If there is only one element, the sum is the value of this element.
- Otherwise, the sum is calculated by adding the first element and the sum of the rest.

Here is the C++ implementation:

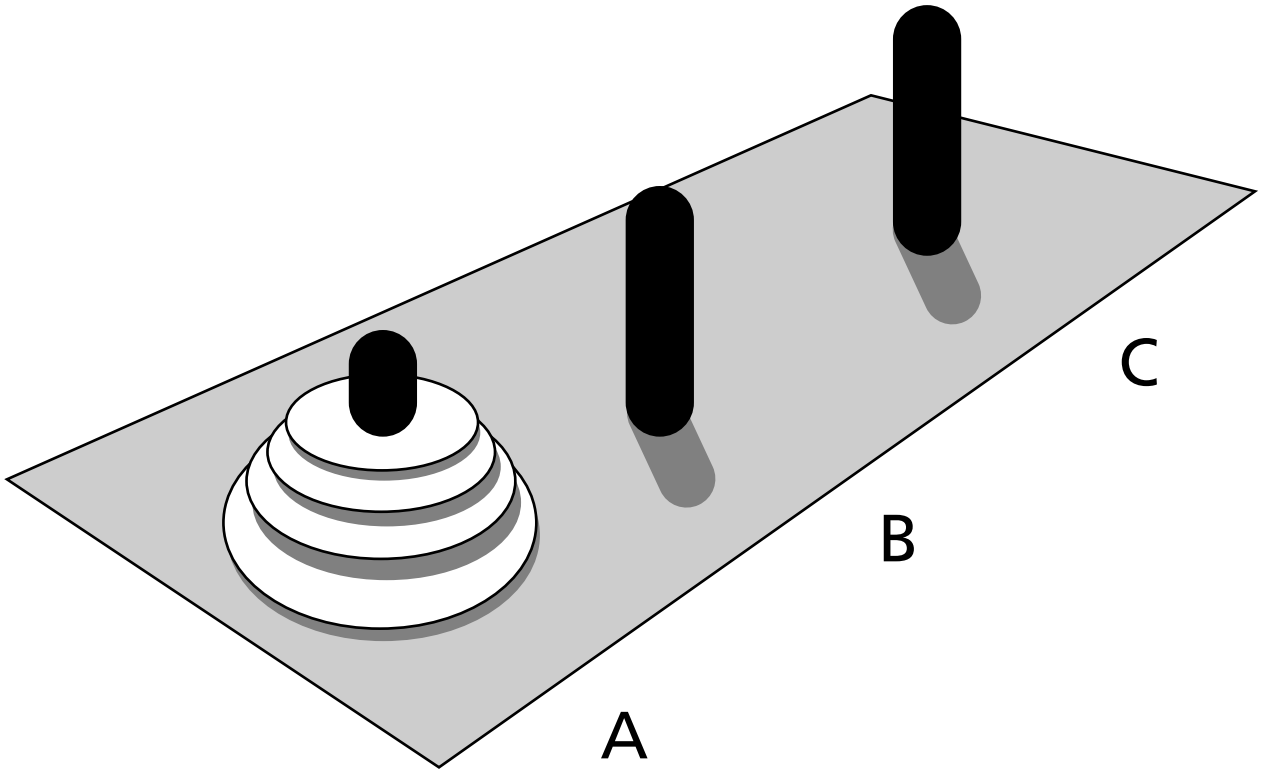
```
int sum(int first, int last, int array[])
{
  if(first == last)
    return array[first];
  /* else */
    return
      (array[first] +
       sum(first+1,last,array));
}
```

Here is another example of the recursive process generated by this function:

```
Sum(1 8 3 2) =

    (1 + Sum(8 3 2)) =

    (1 + (8 + Sum(3 2))) =

    (1 + (8 + (3 + Sum( 2 )))) =

    (1 + (8 + (3 + 2))) =

    (1 + (8 + 6)) =

    (1 + 14) =

    ( 15 )
```

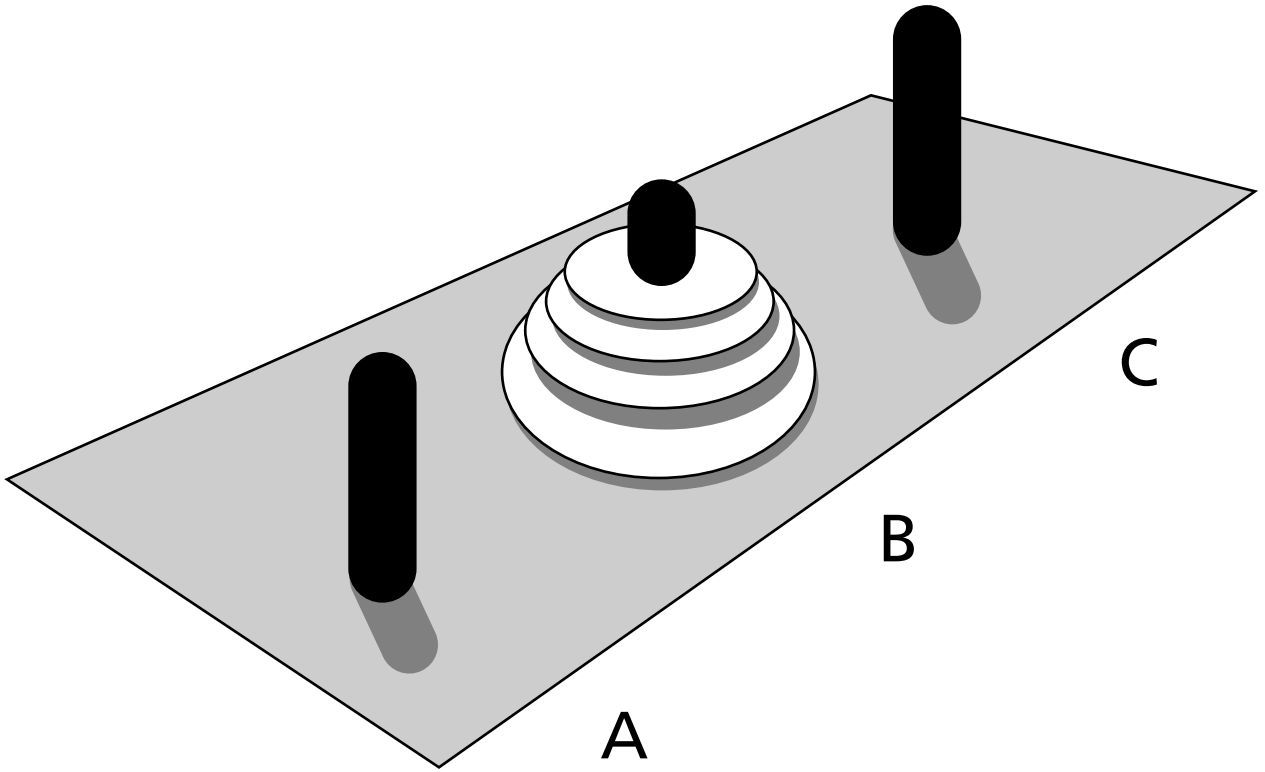
16.3 The Towers of Hanoi

16.3.1 Problem Definition

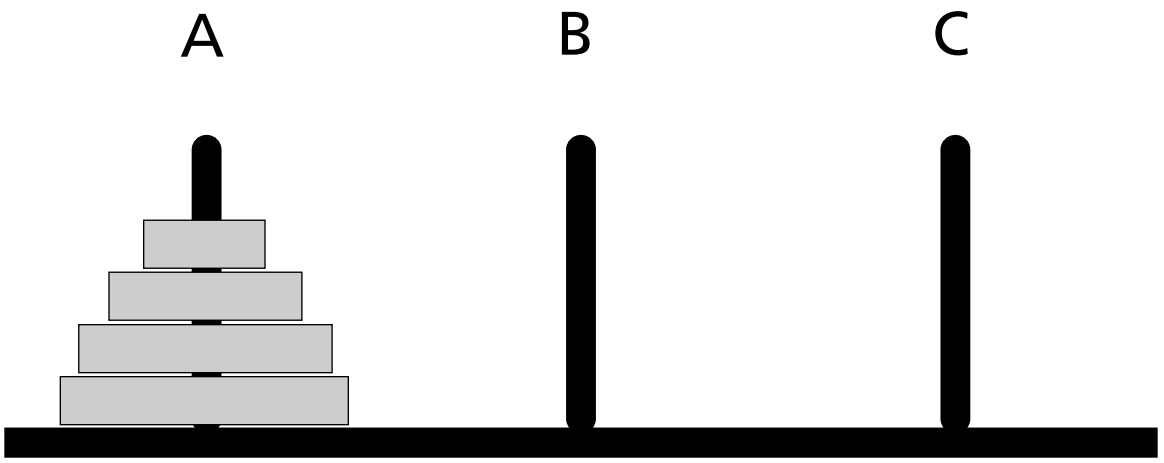


16.3The Towers of Hanoi

16.3.1 Problem Definition



Step 0 — Initial



Initial Setup:

Tower A contains *n* disks of different sizes.
Disks can only go on top of smaller disks (or directly on the board).

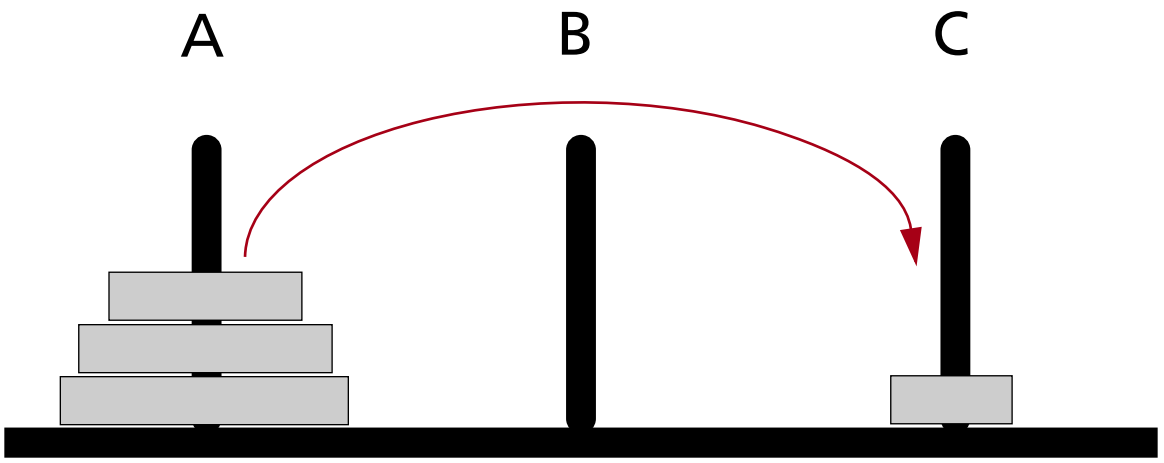
Objec tive:

Move all *n* disks from tower A to tower B.

Moves:

Take the top disk of a tower and move the disk to another tower.
No disk may be put on top of a smaller disk.

Step 1 — Recursion 1



Initial Setup:

Tower A contains *n* disks of different sizes.
Disks can only go on top of smaller disks (or directly on the board).

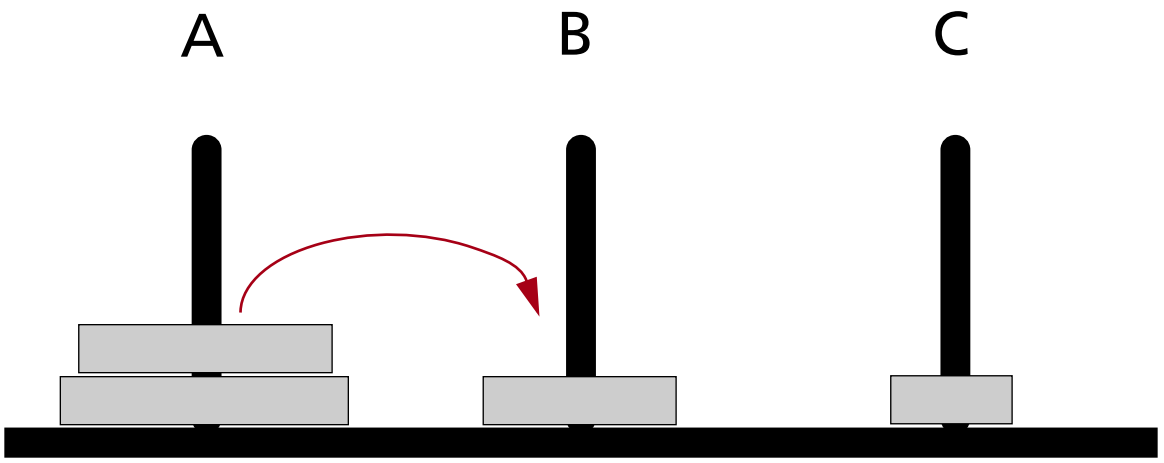
Objec tive:

Move all *n* disks from tower A to tower B.

Moves:

Take the top disk of a tower and move the disk to another tower.
No disk may be put on top of a smaller disk.

Step 2 — Recursion 1



Initial Setup:

Tower A contains *n* disks of different sizes.
Disks can only go on top of smaller disks (or directly on the board).

Objec tive:

Move all *n* disks from tower A to tower B.

Moves:

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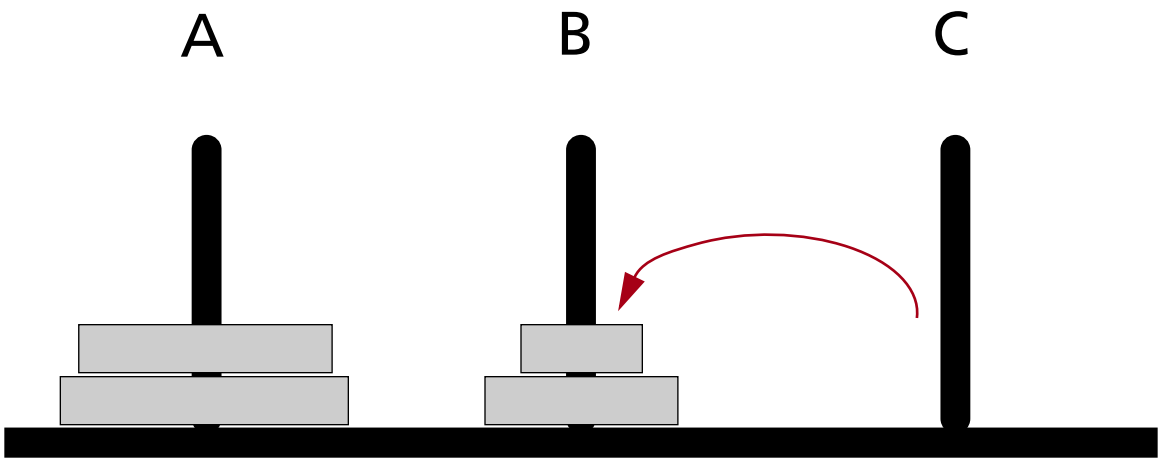
No disk may be put on top of a smaller disk.

Step 3 — Recursion 1



Initial Setup:

Tower A contains *n* disks of different sizes.
Disks can only go on top of smaller disks (or directly on the board).

Objec tive:

Move all *n* disks from tower A to tower B.

Moves:

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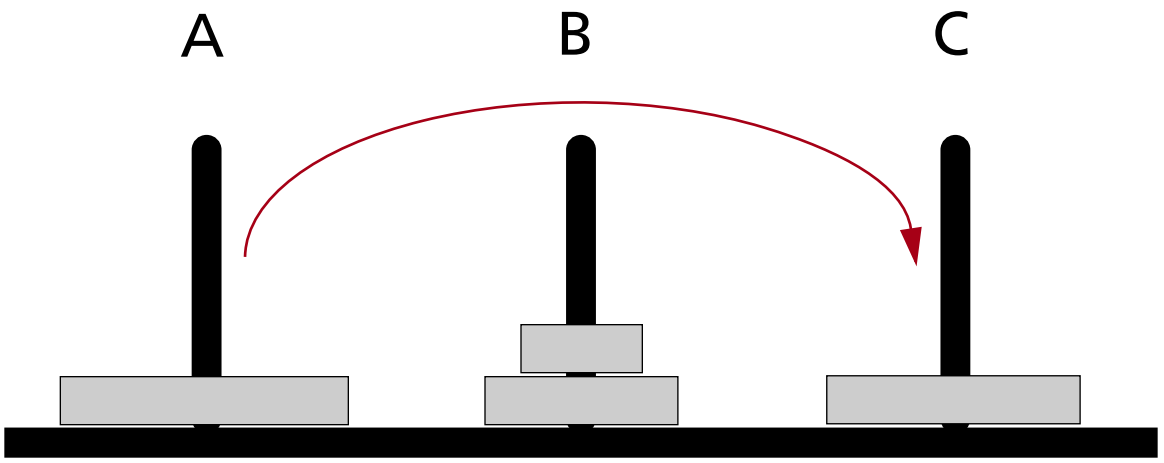
No disk may be put on top of a smaller disk.

Step 4 — Recursion 1



Initial Setup:

Tower A contains *n* disks of different sizes.
Disks can only go on top of smaller disks (or directly on the board).

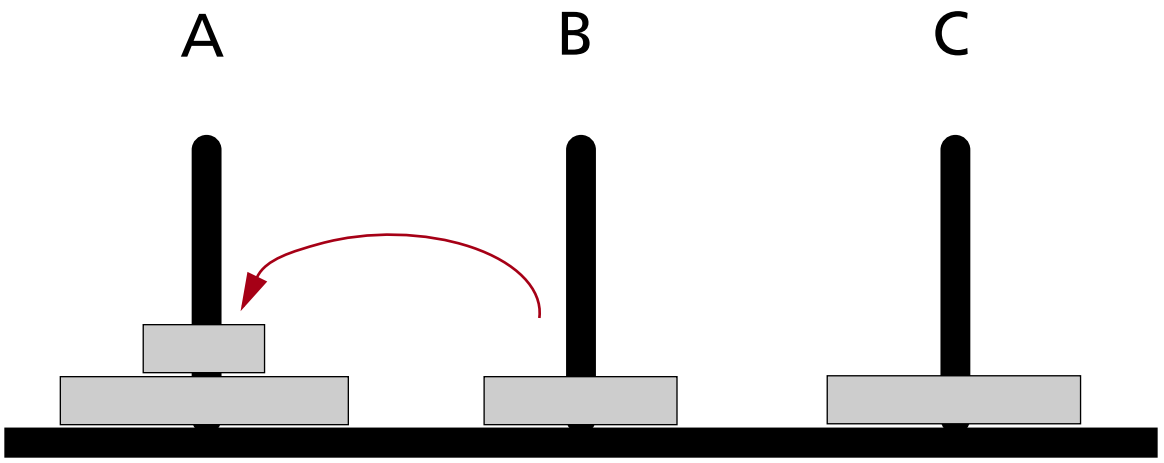
Objec tive:

Move all *n* disks from tower A to tower B.

Moves:

Take the top disk of a tower and move the disk to another tower.
No disk may be put on top of a smaller disk.

Step 5 — Recursion 1



Initial Setup:

Tower A contains *n* disks of different sizes.
Disks can only go on top of smaller disks (or directly on the board).

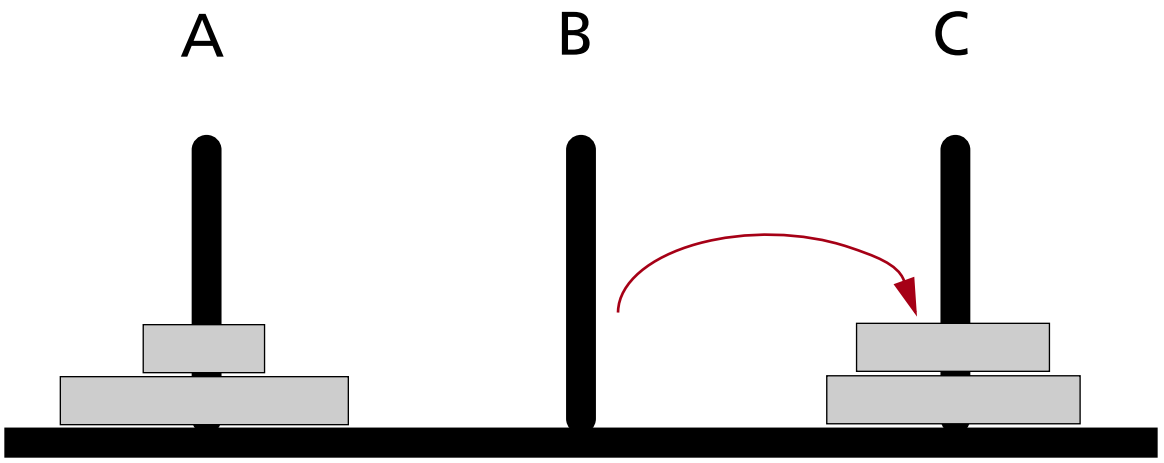
Objec tive:

Move all *n* disks from tower A to tower B.

Moves:

Take the top disk of a tower and move the disk to another tower.
No disk may be put on top of a smaller disk.

Step 6 — Recursion 1



Initial Setup:

Tower A contains *n* disks of different sizes.
Disks can only go on top of smaller disks (or directly on the board).

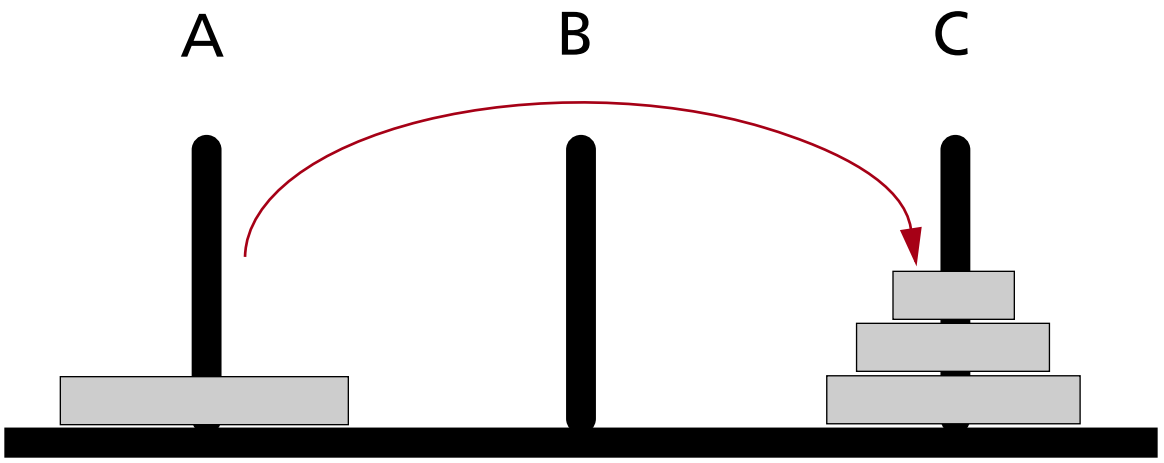
Objec tive:

Move all *n* disks from tower A to tower B.

Moves:

Take the top disk of a tower and move the disk to another tower.
No disk may be put on top of a smaller disk.

Step 7 — Recursion 1



Initial Setup:

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Disks can only go on top of smaller disks (or directly on the board).

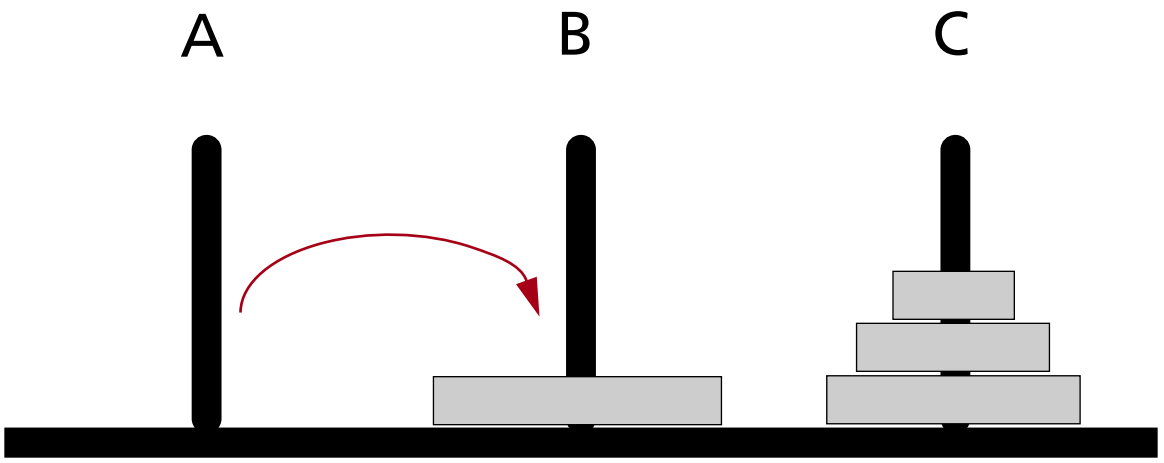
Objec tive:

Move all *n* disks from tower A to tower B.

Moves:

Take the top disk of a tower and move the disk to another tower.
No disk may be put on top of a smaller disk.

Step 8 — Move



Initial Setup:

Tower A contains *n* disks of different sizes.
Disks can only go on top of smaller disks (or directly on the board).

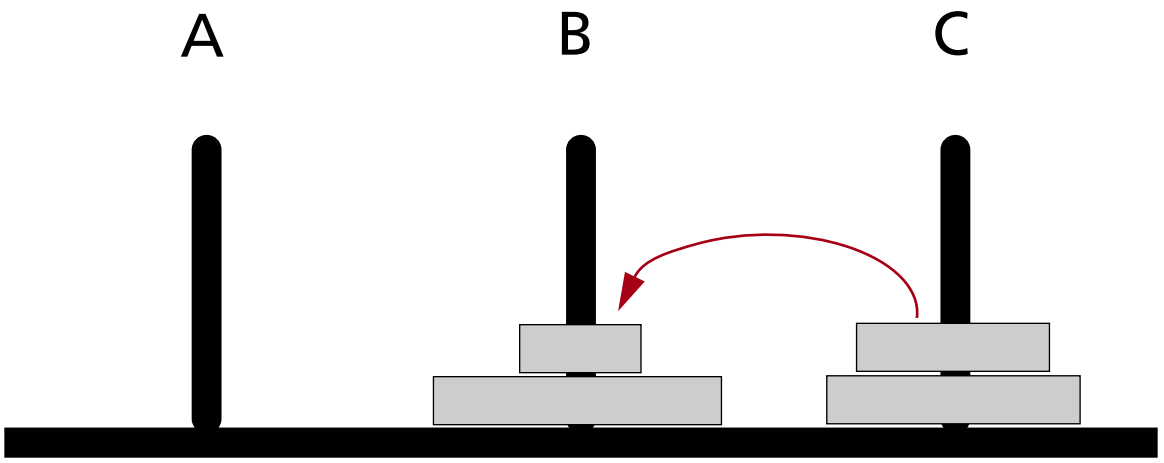
Objec tive:

Move all *n* disks from tower A to tower B.

Moves:

Take the top disk of a tower and move the disk to another tower.
No disk may be put on top of a smaller disk.

Step 9 — Recursion 2



Initial Setup:

Tower A contains *n* disks of different sizes.
Disks can only go on top of smaller disks (or directly on the board).

Objec tive:

Move all *n* disks from tower A to tower B.

Moves:

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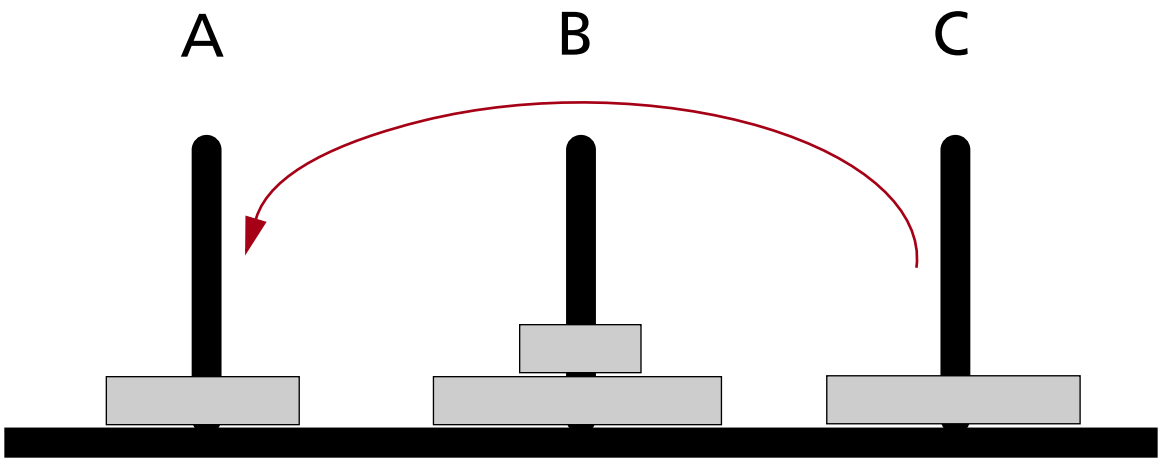
No disk may be put on top of a smaller disk.

Step 10 — Recursion 2



Initial Setup:

Tower A contains *n* disks of different sizes.
Disks can only go on top of smaller disks (or directly on the board).

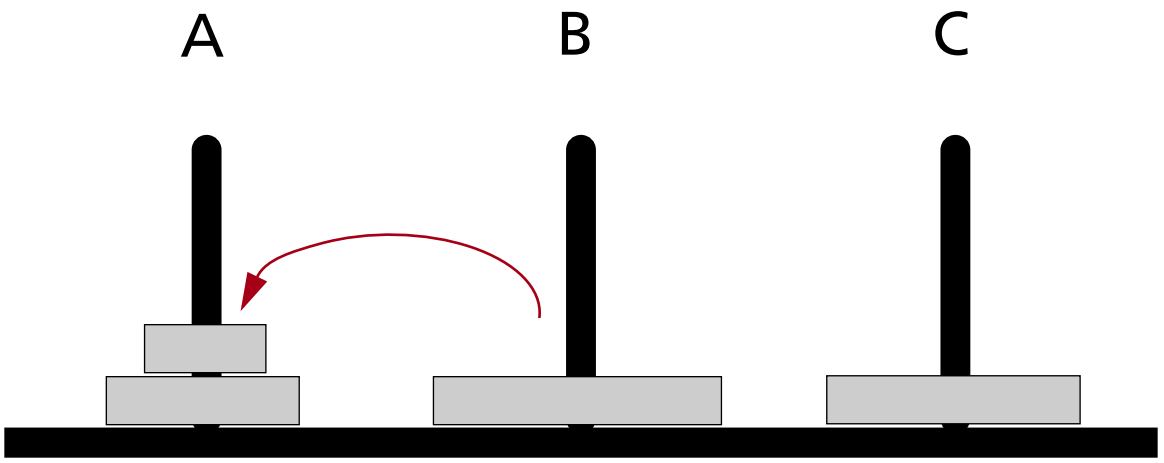
Objec tive:

Move all *n* disks from tower A to tower B.

Moves:

Take the top disk of a tower and move the disk to another tower.
No disk may be put on top of a smaller disk.

Step 11 — Recursion 2



Initial Setup:

Tower A contains *n* disks of different sizes.
Disks can only go on top of smaller disks (or directly on the board).

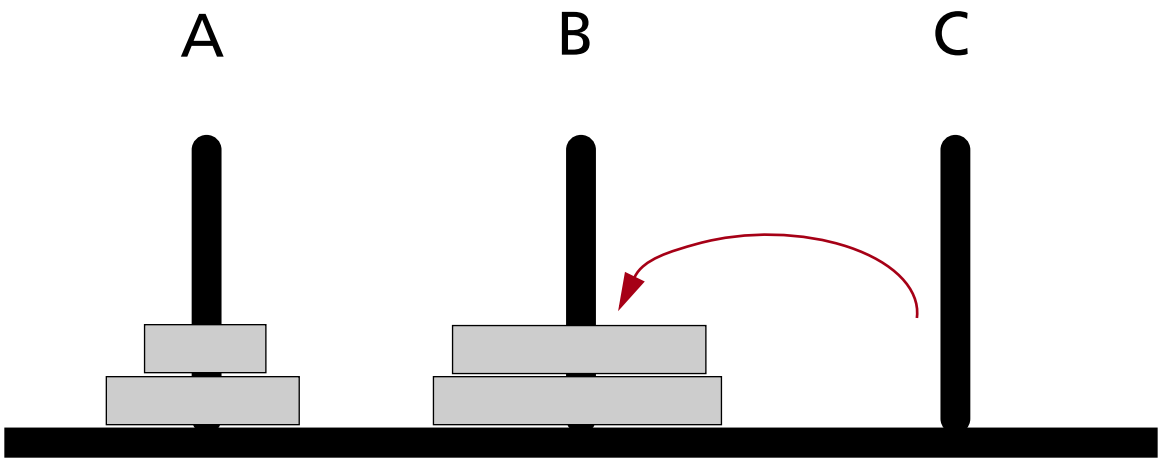
Objec tive:

Move all *n* disks from tower A to tower B.

Moves:

Take the top disk of a tower and move the disk to another tower.
No disk may be put on top of a smaller disk.

Step 12 — Recursion 2



Initial Setup:

Tower A contains *n* disks of different sizes.
Disks can only go on top of smaller disks (or directly on the board).

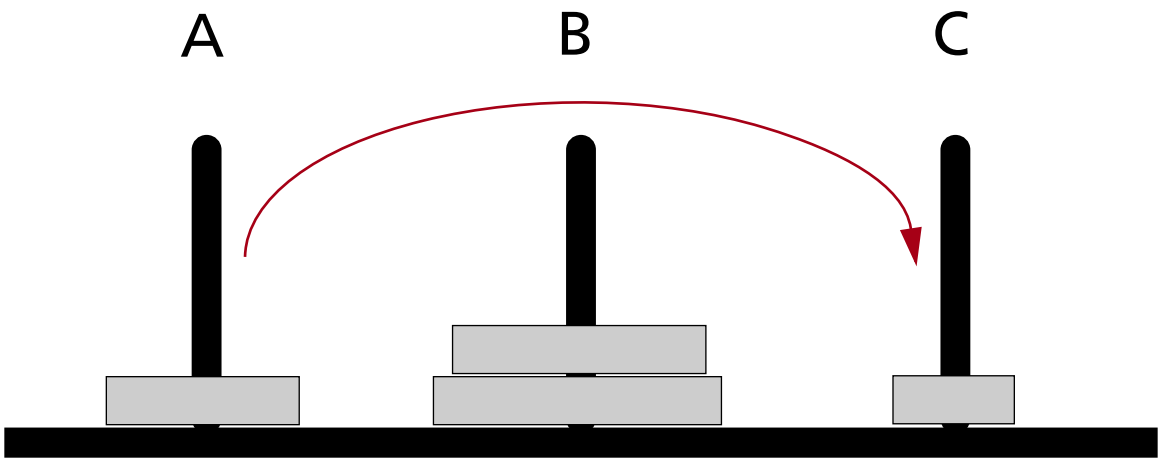
Objec tive:

Move all *n* disks from tower A to tower B.

Moves:

Take the top disk of a tower and move the disk to another tower.
No disk may be put on top of a smaller disk.

Step 13 — Recursion 2



Initial Setup:

Tower A contains *n* disks of different sizes.
Disks can only go on top of smaller disks (or directly on the board).

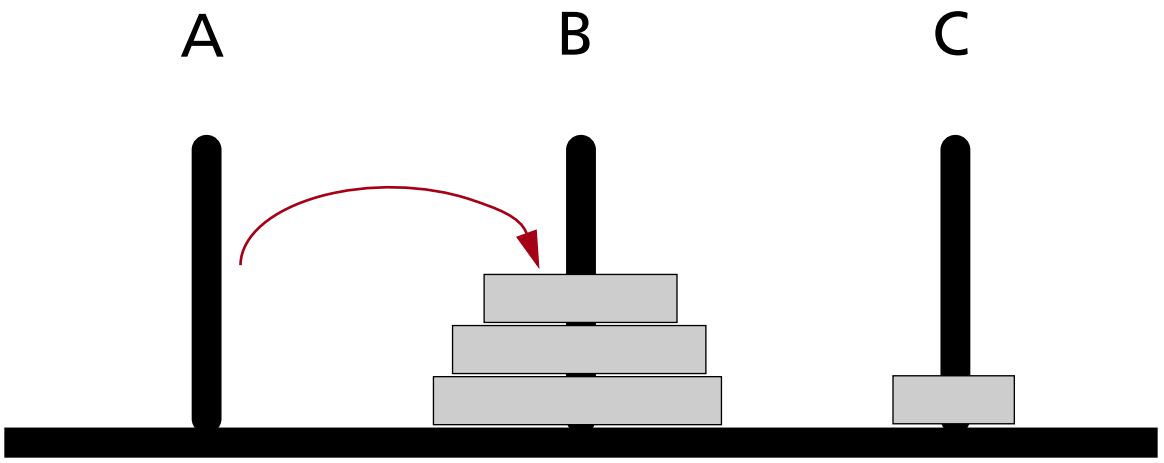
Objec tive:

Move all *n* disks from tower A to tower B.

Moves:

Take the top disk of a tower and move the disk to another tower.
No disk may be put on top of a smaller disk.

Step 14 — Recursion 2



Initial Setup:

Tower A contains *n* disks of different sizes.
Disks can only go on top of smaller disks (or directly on the board).

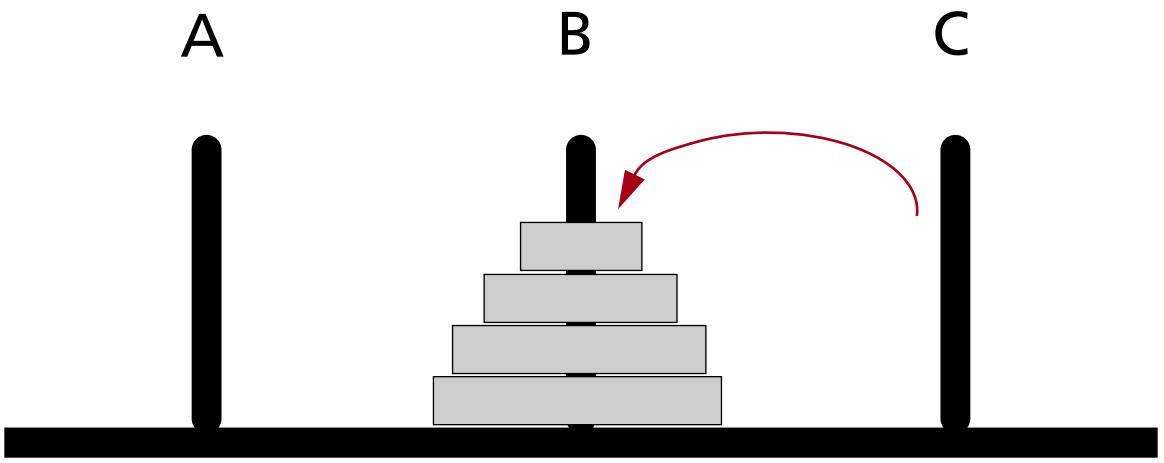
Objec tive:

Move all *n* disks from tower A to tower B.

Moves:

Take the top disk of a tower and move the disk to another tower.
No disk may be put on top of a smaller disk.

Step 15 — Recursion 2



Initial Setup:

Tower A contains *n* disks of different sizes.
Disks can only go on top of smaller disks (or directly on the board).

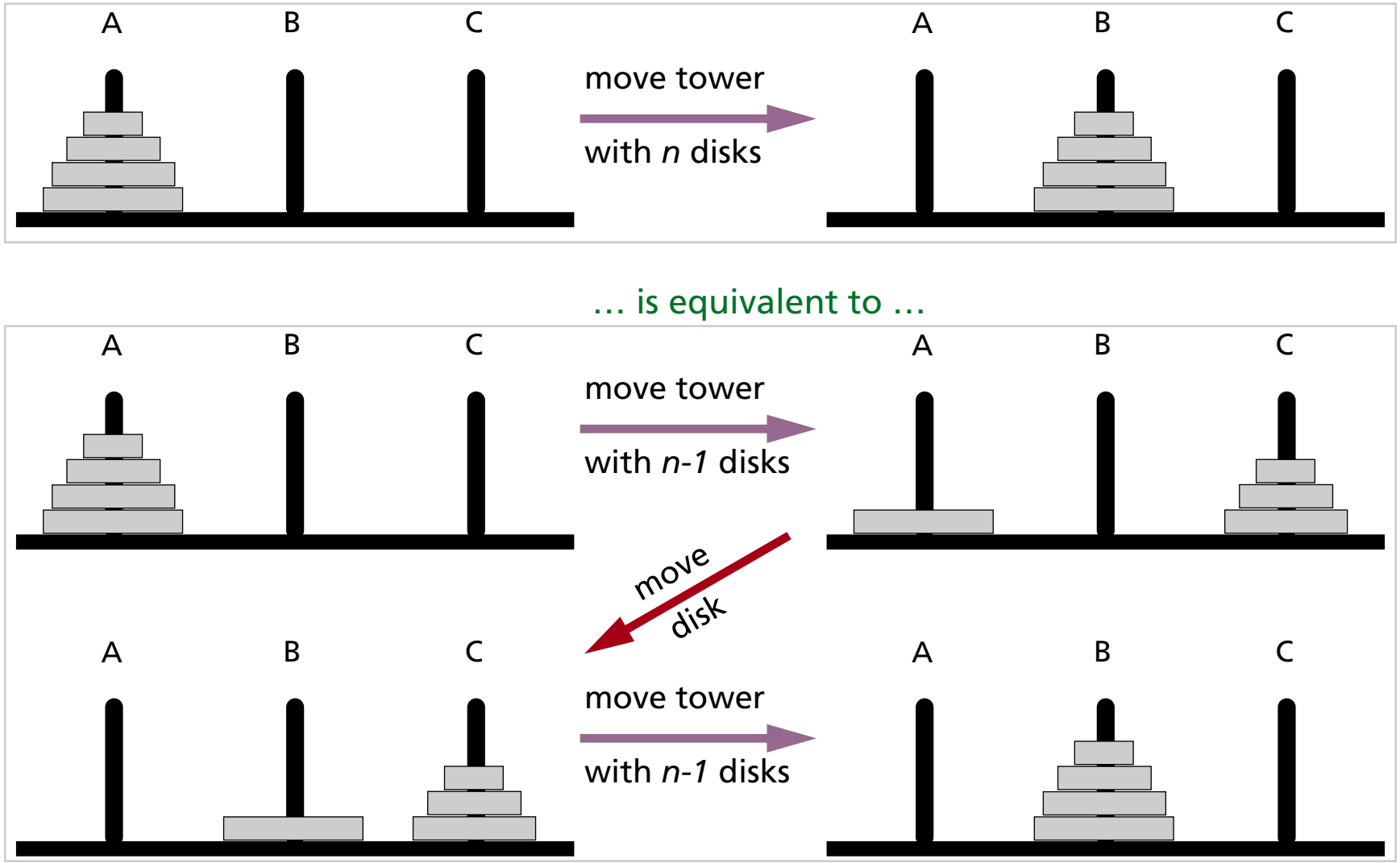
Objec tive:

Move all *n* disks from tower A to tower B.

Moves:

Take the top disk of a tower and move the disk to another tower.
No disk may be put on top of a smaller disk.

16.3.2 Ideas for a Recursive Solution



16.3.3 A Recursive Tower-of-Hanoi Algorithm

The Main Algorithm:

```
int main()
{
   int n;

   cout << "How many disks? ";
   cin >> n;

   if( n > 0)
      moveTower(n, 'A', 'B', 'C');

   return 0;
}
```

Exponential problem: *n* disks ---> 2^n-1 moves

Algorithm output:

How many disks? 4

Move top disk from tower A to tower C.
Move top disk from tower A to tower B.
Move top disk from tower C to tower B.
Move top disk from tower A to tower C.
Move top disk from tower B to tower A.
Move top disk from tower B to tower C.
Move top disk from tower A to tower C.
Move top disk from tower A to tower B.
Move top disk from tower C to tower B.
Move top disk from tower C to tower A.
Move top disk from tower B to tower A.
Move top disk from tower C to tower B.
Move top disk from tower A to tower C.
Move top disk from tower A to tower B.
Move top disk from tower C to tower B.

```
void moveTower(int n, char from, char to,
                      char temp)
{
  if( n==1 )
    moveDisk(from, to);
  else {
    moveTower(n-1, from, temp, to);
    moveDisk(from, to);
    moveTower(n-1, temp, to, from);
  }
}

void moveDisk(char from, char to)
{
  cout << “Move top disk from tower “ << from
  cout << “ to tower “ << to << “.” << endl;
}
```

16.4 References

- G. Blank and R. Barnes, *The Universal Machine*, Boston, MA: WCB/ McGraw-Hill, 1998. Chapter 9.8.
- H. Schildt, C++ from the Ground Up, McGraw-Hill, Berkeley, CA, 1998. Chapter 7.