Binomial Heaps

Outline for this Week

- **Binomial Heaps (Today)**
  - A simple, flexible, and versatile priority queue.
- **Lazy Binomial Heaps (Today)**
  - A powerful building block for designing advanced data structures.
- **Fibonacci Heaps (Wednesday)**
  - A heavyweight and theoretically excellent priority queue.

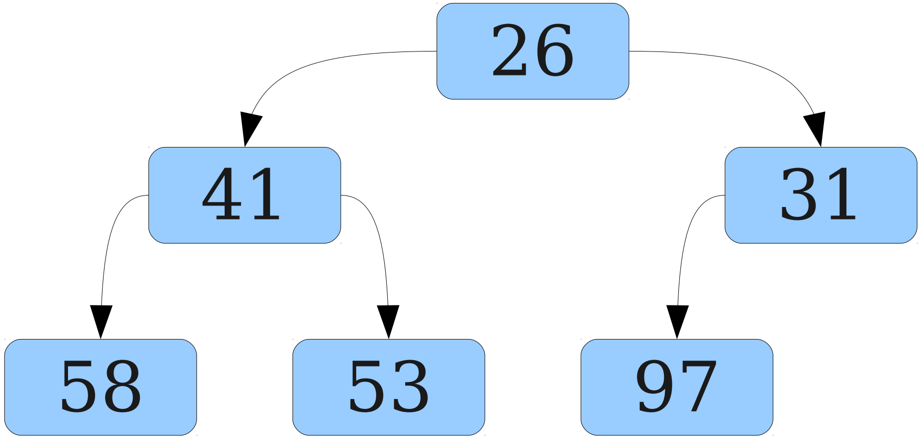
Review: Priority Queues

Priority Queues

- A **priority queue** is a data structure that stores a set of elements annotated with *keys* and allows efficient extraction of the element with the least key.
- More concretely, supports these operations:
  - *pq*.***enqueue***(*v*, *k*), which enqueues element *v* with key *k*;
  - *pq*.***find-min***(), which returns the element with the least key; and
  - *pq*.***extract-min***(), which removes and returns the element with the least key,

Binary Heaps

- Priority queues are frequently implemented as **binary heaps**.
- ***enqueue*** and ***extract-min*** run in time O(log *n*); ***find-min*** runs in time O(1).
- We're not going to cover binary heaps this quarter; I assume you've seen them before.



Priority Queues in Practice

- Many graph algorithms directly rely priority queues supporting extra operations:
  - ***meld***(pq_1, pq_2): Destroy pq_1 and pq_2 and combine their elements into a single priority queue.
  - *pq*.***decrease-key***(v, k'): Given a pointer to element v already in the queue, lower its key to have new value k'.
  - *pq*.***add-to-all***(Δk): Add Δk to the keys of each element in the priority queue (typically used with ***meld***).
- In lecture, we'll cover binomial heaps to efficiently support ***meld*** and Fibonacci heaps to efficiently support ***meld*** and ***decrease-key***.
- After the TAs ensure that it's not too hard to do so, you'll design a priority queue supporting efficient ***meld*** and ***add-to-all*** on the problem set.

Meldable Priority Queues

- A priority queue supporting the ***meld*** operation is called a **meldable priority queue**.
- ***meld***(pq_1, pq_2) destructively modifies pq_1 and pq_2 and produces a new priority queue containing all elements of pq_1 and pq_2.

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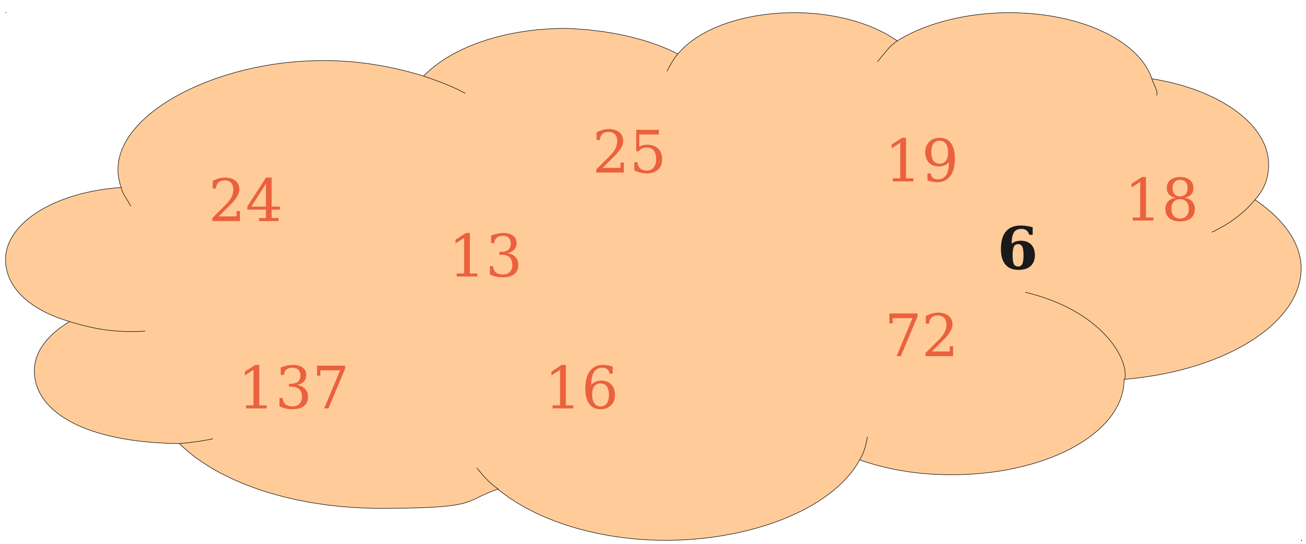
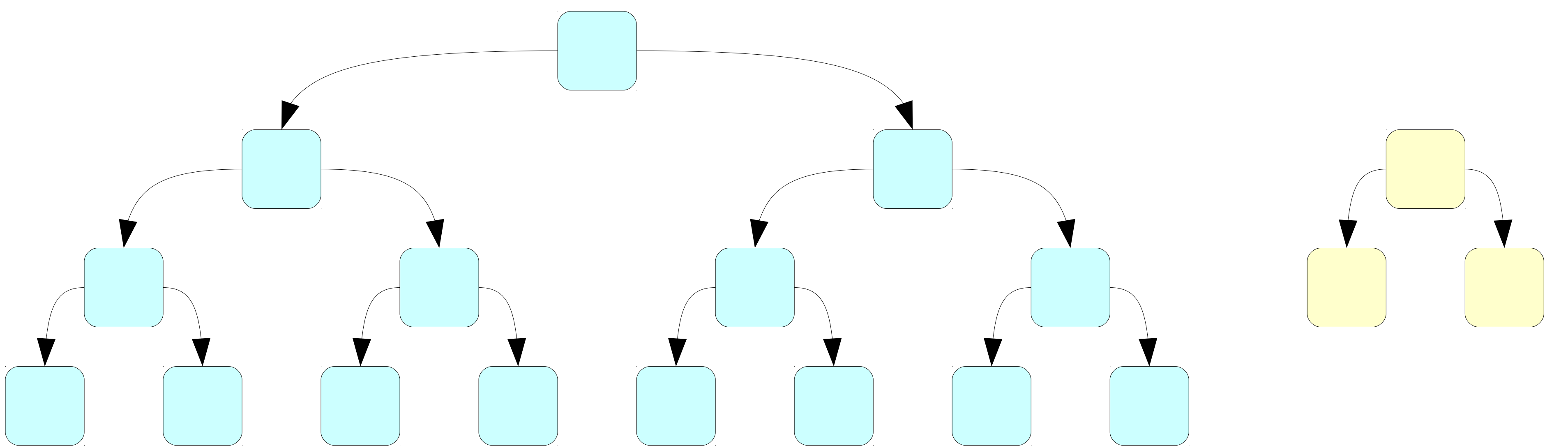


Efficiently Meldable Queues

- Standard binary heaps do not efficiently support ***meld***.
- **Intuition**: Binary heaps are complete binary trees, and two complete binary trees cannot easily be linked to one another.



Binomial Heaps

- The **binomial heap** is an efficient priority queue data structure that supports efficient melding.
- We'll study binomial heaps for several reasons:
  - Implementation and intuition is totally different than binary heaps.
  - Used as a building block in other data structures (Fibonacci heaps, soft heaps, etc.)
  - Has a beautiful intuition; similar ideas can be used to produce other data structures.

The Intuition: **Binary Arithmetic**

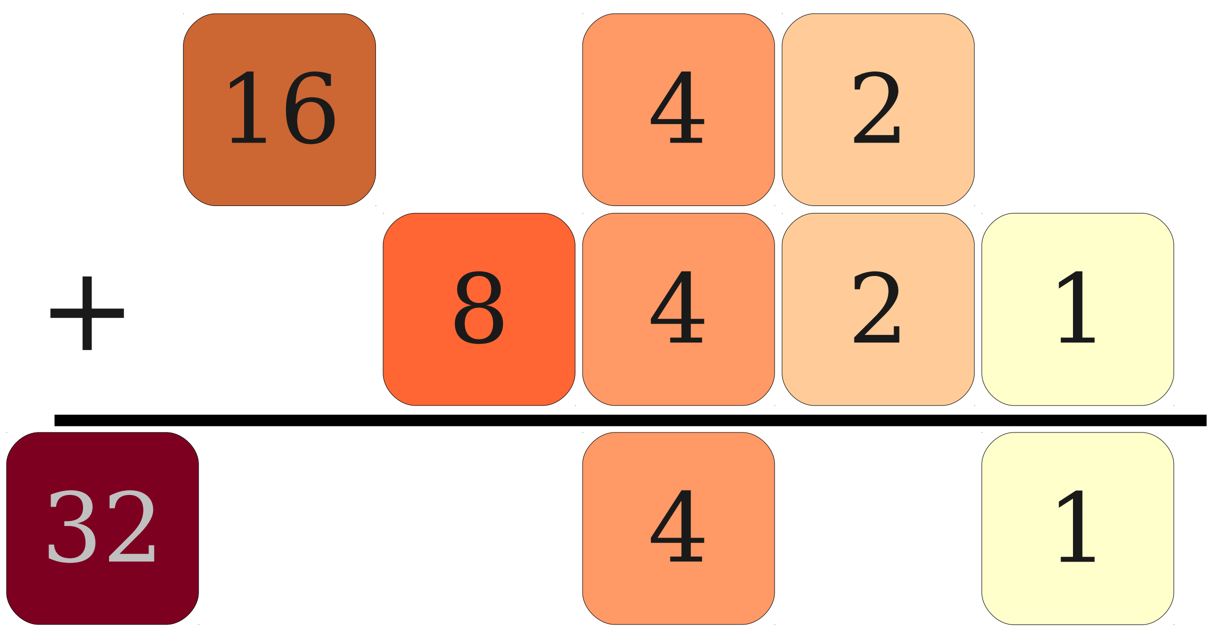
Adding Binary Numbers

- Given the binary representations of two numbers n and m, we can add those numbers in time $\Theta(\max\{\log m, \log n\})$.

	1	0	1	1	0
+		1	1	1	1

A Different Intuition

- Represent *n* and *m* as a collection of “packets” whose sizes are powers of two.
- Adding together *n* and *m* can then be thought of as combining the packets together, eliminating duplicates

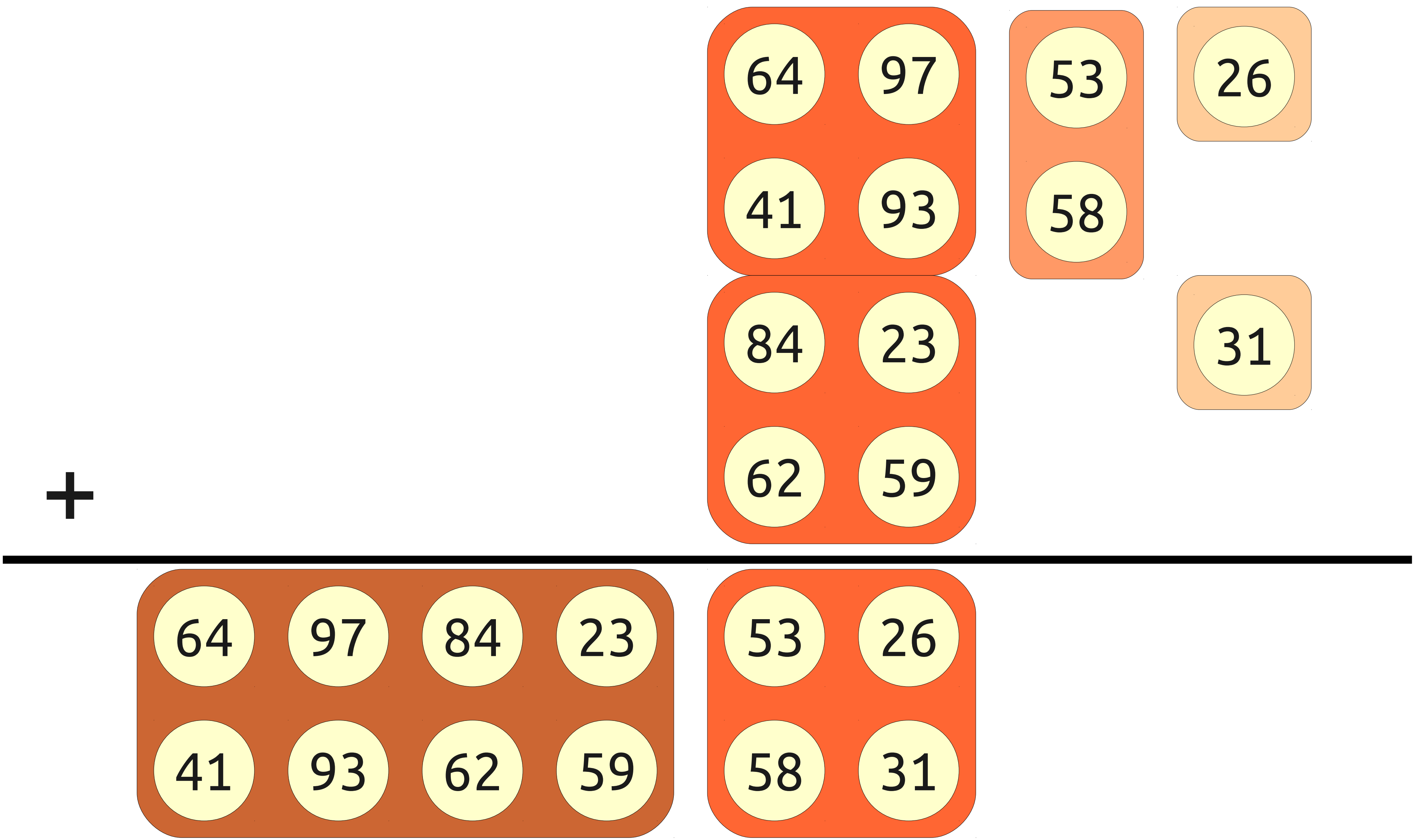


Why This Works

- In order for this arithmetic procedure to work efficiently, the packets must obey the following properties:
  - The packets must be stored in ascending/descending order of size.
  - The packets must be stored such that there are no two packets of the same size.
  - Two packets of the same size must be efficiently “fusable” into a single packet.

Building a Priority Queue

- **Idea:** Adapt this approach to build a priority queue.
- Store elements in the priority queue in "packets" whose sizes are powers of two.
- Store packets in ascending size order.
- We'll choose a representation of a packet so that two packets of the same size can easily be fused together.


64 97
41 93
53
58
26
84 23
62 59
31
+
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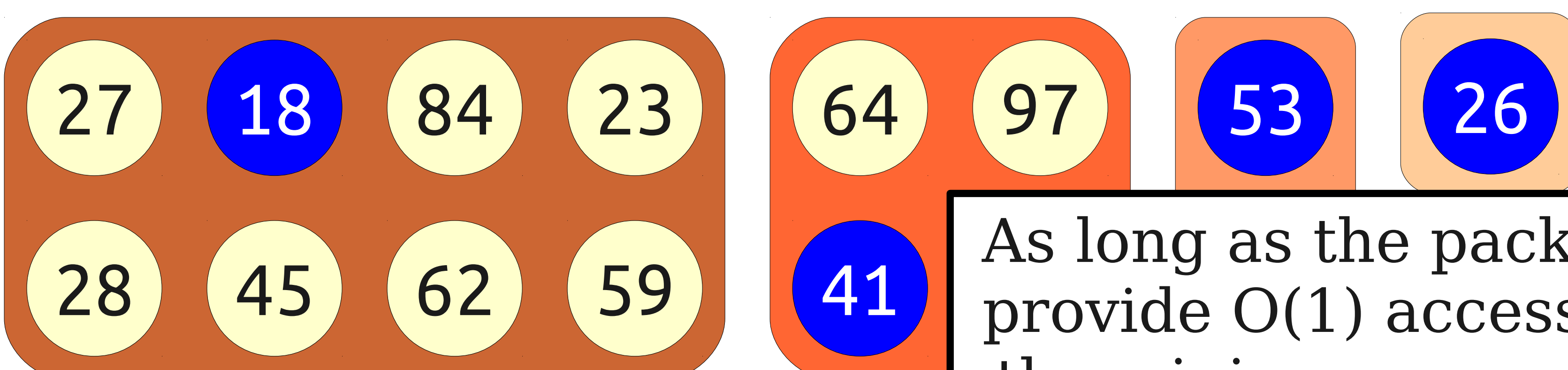
Building a Priority Queue

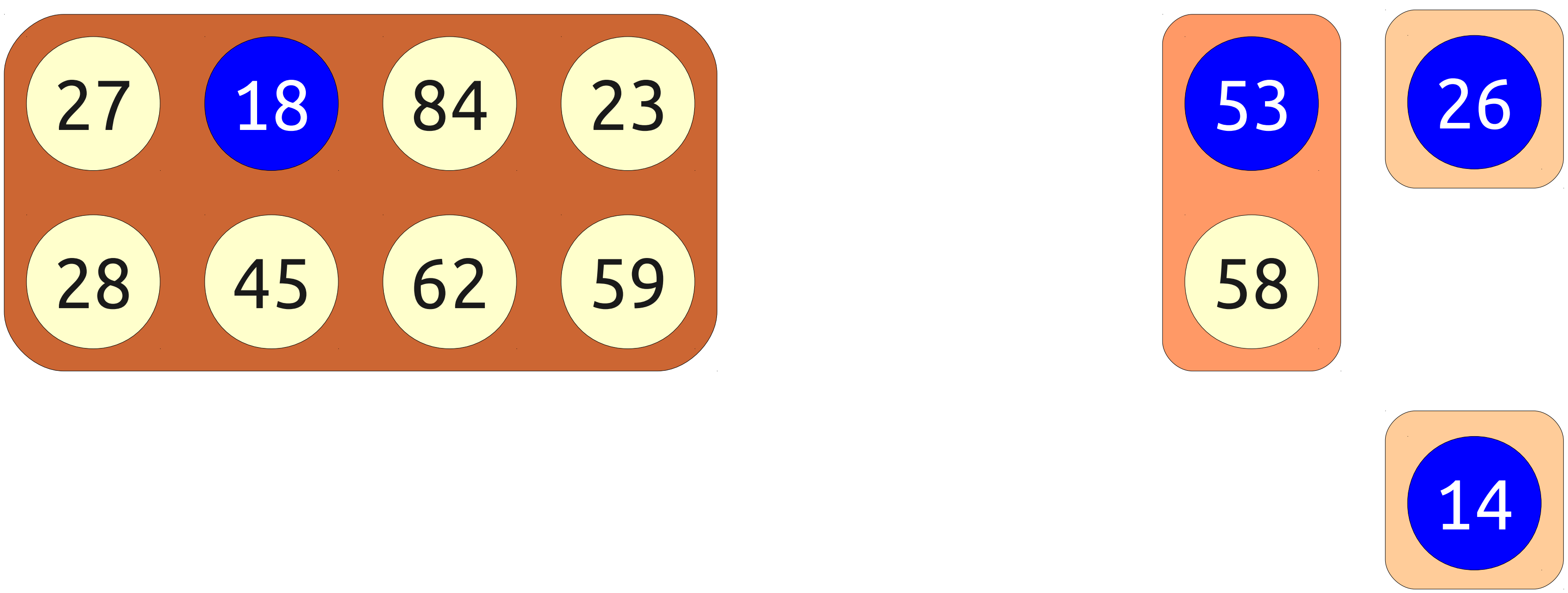
- What properties must our packets have?
  - Sizes must be powers of two.
  - Can efficiently fuse packets of the same size.



As long as the packets provide O(1) access to the minimum, we can execute ***find-min*** in time O(log *n*).

Inserting into the Queue

- If we can efficiently meld two priority queues, we can efficiently enqueue elements to the queue.
- **Idea:** Meld together the queue and a new queue with a single packet.



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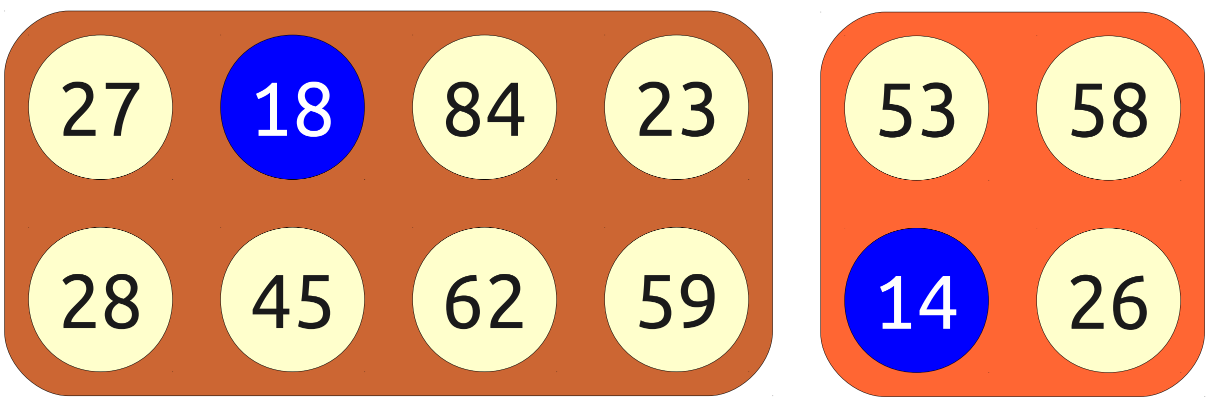


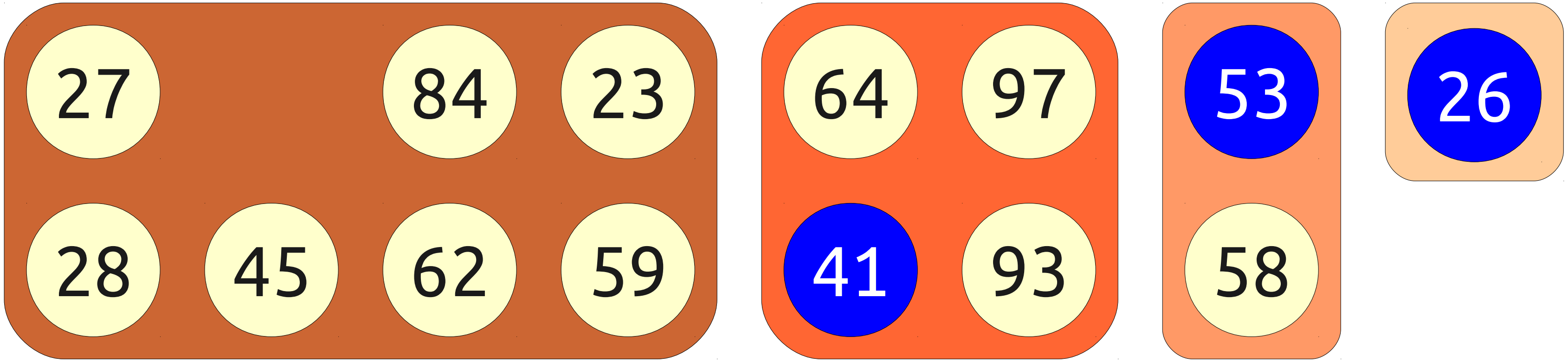
Time required:
O(log n) fuses.

Fracturing Packets

- If we have a packet with 2^k elements in it and remove a single element, we are left with $2^k - 1$ remaining elements.
- **Fun fact**: $2^k - 1 = 1 + 2 + 4 + \ldots + 2^{k-1}$.
- **Idea**: "Fracture" the packet into $k - 1$ smaller packets, then add them back in.

Fracturing Packets

- We can ***extract-min*** by fracturing the packet containing the minimum and adding the fragments back in.



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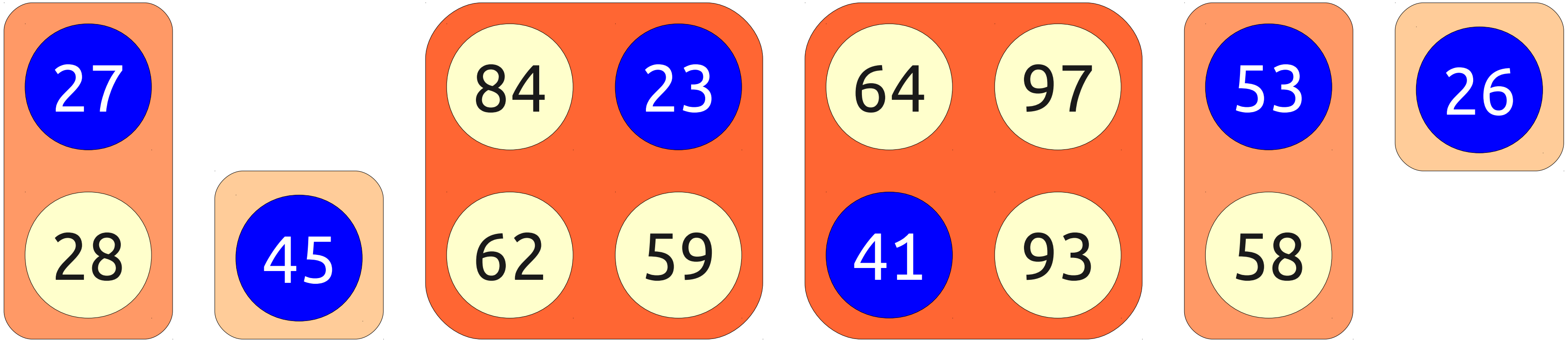
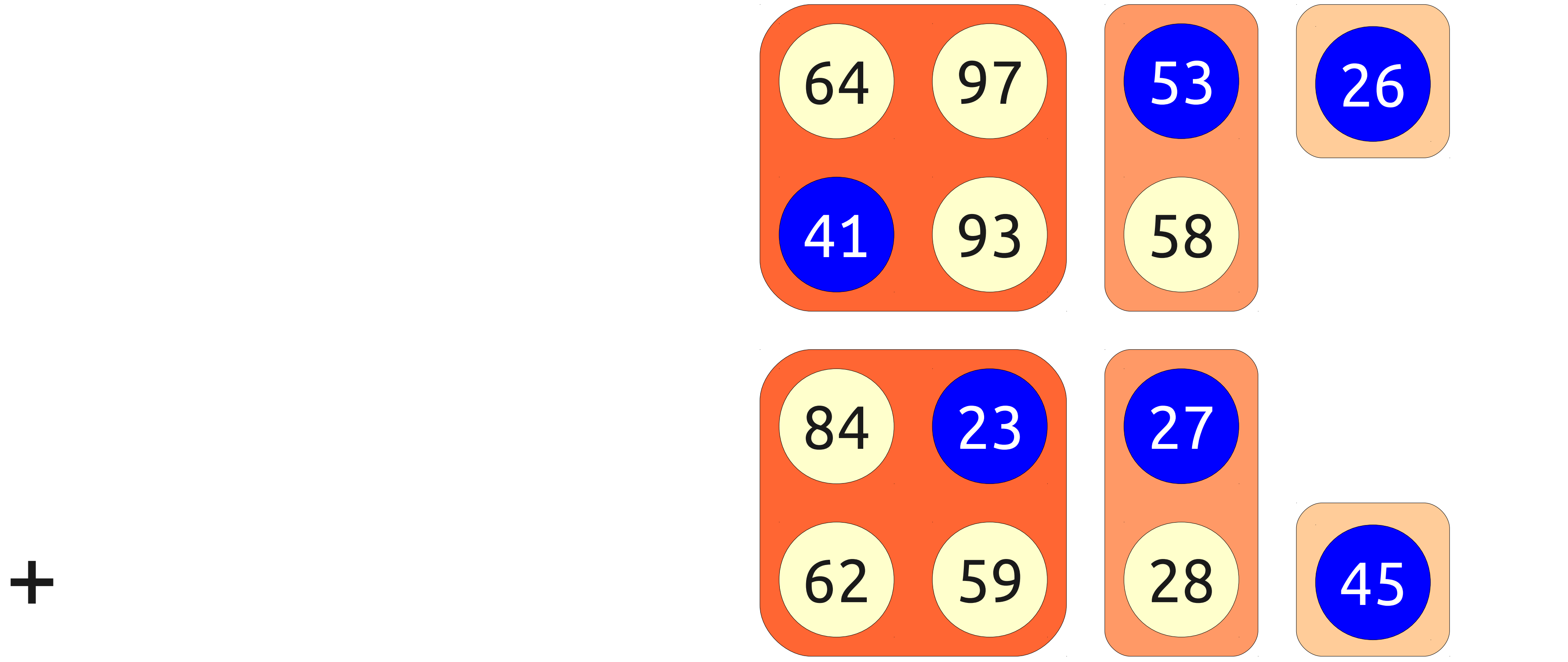


Fracturing Packets

- We can ***extract-min*** by fracturing the packet containing the minimum and adding the fragments back in.
- Runtime is O(log *n*) fuses in ***meld***, plus fragment cost.



Building a Priority Queue

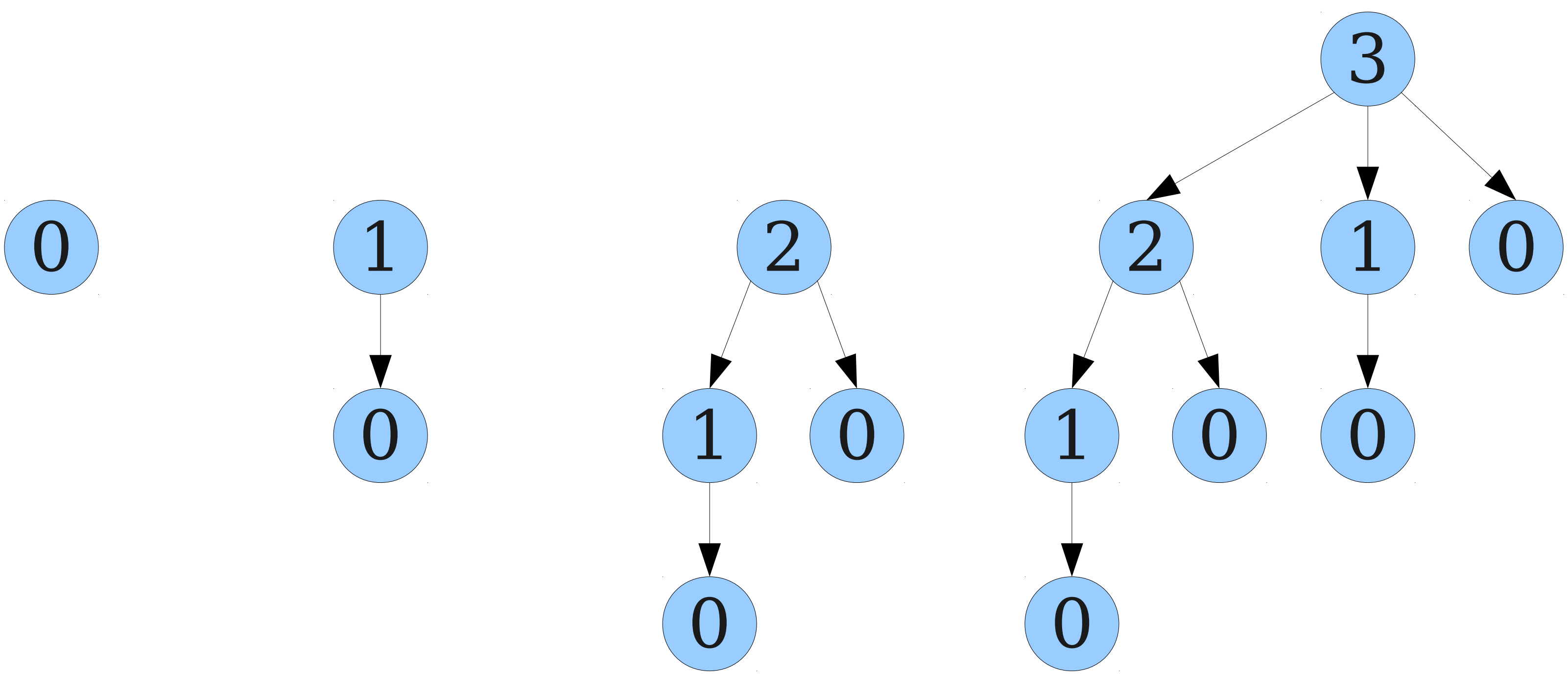
- What properties must our packets have?
  - Size must be a power of two.
  - Can efficiently fuse packets of the same size.
  - Can efficiently find the minimum element of each packet.
  - Can efficiently “fracture” a packet of 2^k nodes into packets of 1, 2, 4, 8, …, 2^{k-1} nodes.
- What representation of packets will give us these properties?

Binomial Trees

- A **binomial tree of order *k*** is a type of tree recursively defined as follows:

  A binomial tree of order *k* is a single node whose children are binomial trees of order 0, 1, 2, …, *k* – 1.

- Here are the first few binomial trees:



Binomial Trees

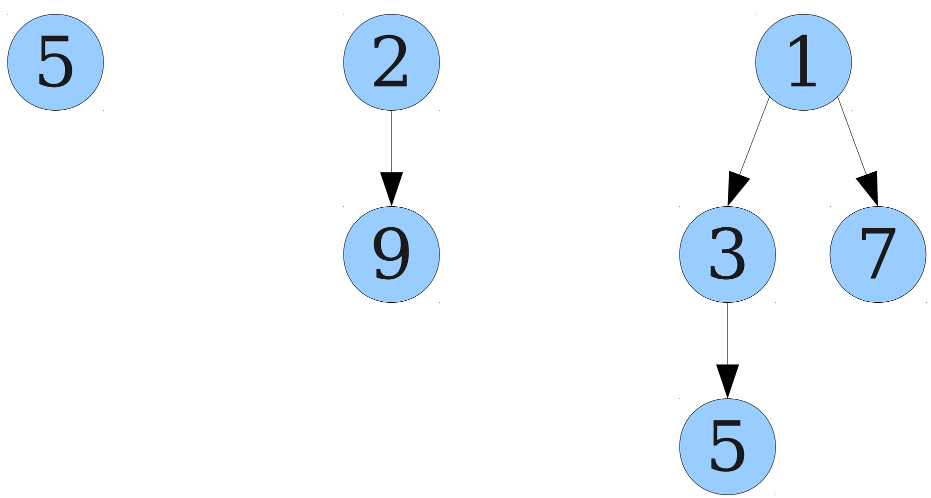
- ***Theorem:*** A binomial tree of order k has exactly 2^k nodes.
- ***Proof:*** Induction on k. Assuming that binomial trees of orders 0, 1, 2, …, k – 1 have 2^0, 2^1, 2^2, …, 2^{k-1} nodes, then then number of nodes in an order-k binomial tree is

$$2^0 + 2^1 + \ldots + 2^{k-1} + 1 = 2^k - 1 + 1 = 2^k$$

So the claim holds for k as well. ■

Binomial Trees

- A **heap-ordered binomial tree** is a binomial tree whose nodes obey the heap property: all nodes are less than or equal to their descendants.
- We will use heap-ordered binomial trees to implement our "packets."



Binomial Trees

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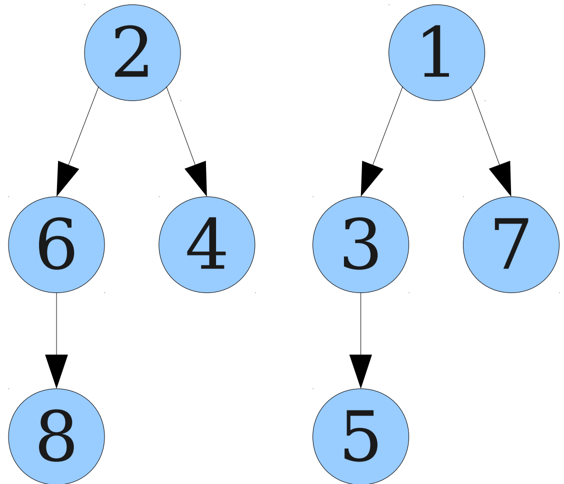

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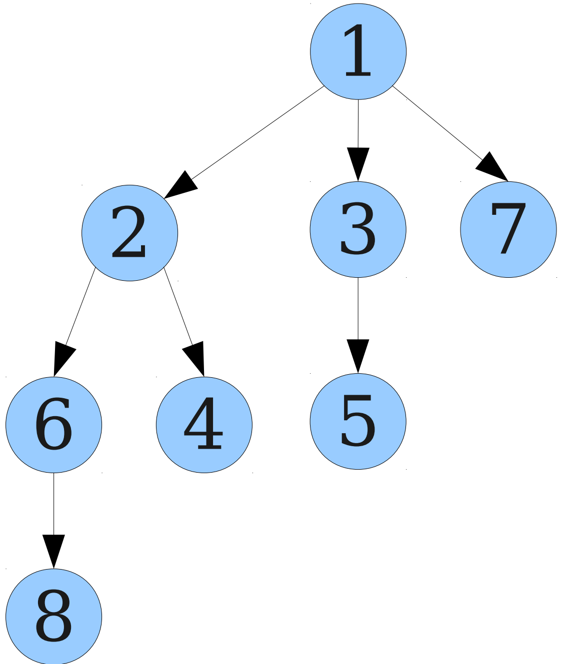

Make the binomial tree with the larger root the first child of the tree with the smaller root.

Binomial Trees

- What properties must our packets have?
  - Size must be a power of two. ✓
  - Can efficiently fuse packets of the same size. ✓
  - Can efficiently find the minimum element of each packet. ✓
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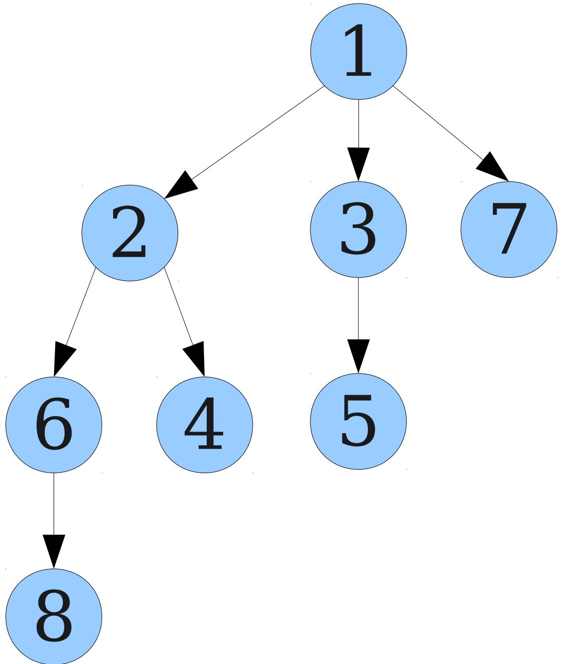

Binomial Trees

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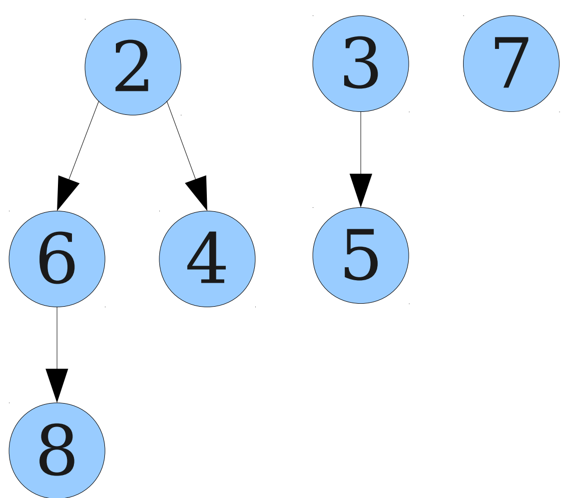


The Binomial Heap

- A **binomial heap** is a collection of heap-ordered binomial trees stored in ascending order of size.
- Operations defined as follows:
  - ***meld***(pq_1, pq_2): Use addition to combine all the trees.
    - Fuses O(log n) trees. Total time: O(log n).
  - pq.***enqueue***(v, k): Meld pq and a singleton heap of (v, k).
    - Total time: O(log n).
  - pq.***find-min***(): Find the minimum of all tree roots.
    - Total time: O(log n).
  - pq.***extract-min***(): Find the min, delete the tree root, then meld together the queue and the exposed children.
    - Total time: O(log n).

Time-Out for Announcements!

Office Hours Update

- Keith's office hours are now moved to Gates 178 going forward – looks like we didn't actually have Hewlett 201 after lecture. ☺
- Thursday office hours changed from 7:30PM – 9:30PM, location TBA.
- As always, feel free to email us with questions!

Problem Set Two Graded

- Problem Set Two has been graded; will be returned at end of lecture.
- Rough solution sketches available up front!

Problem Set Three Clarification

- Many of you have questions about Q2 on Problem Set Three.
- For parts (iii) and (iv), assume the following:
  - The basic data structure can be constructed in worst-case time O(n).
  - The cost of a cut is worst-case O(min{$|T_1|$, $|T_2|$}).
- You don't need to justify these facts. We're mostly interested in seeing your amortized analyses.

Your Questions

“What's a popular data structure in place of map for military purposes, where guaranteed time of operations are required?”

Red/black trees are the gold standard here – they've got excellent worst-case performance and support fast insertions and deletions.

Hash tables have *expected* O(1) operations, but that requires good hash functions. Search “HashDoS” for an attack on many programming languages' implementations of hash tables.

"How do you determine out of how many fewer points a problem set will be worth for people working alone vs. in pairs? Are you happy with how the optional pairs system has worked thus far?"

For PS1, about 25% the class worked in pairs.
For PS2, about 50% of the class worked in pairs.

I'm hoping to encourage people to work in pairs without punishing people who choose not to. I'm still tuning the buffer amount.

"Can you write a CS-themed musical for us?"

I'm thinking *Les Miserables* could be adapted for CS.
Some sample songs:

"Server in the Cloud"

"Red and Black"

"Do you Hear the Balanced Tree?"

Back to CS166!

Analyzing Insertions

- Each ***enqueue*** into a binomial heap takes time O(log *n*), since we have to meld the new node into the rest of the trees.
- However, it turns out that the amortized cost of an insertion is lower in the case where we do a series of *n* insertions.

Adding One

- Suppose we want to execute n++ on the binary representation of n.
- Do the following:
  - Find the longest span of 1's at the right side of n.
  - Flip those 1's to 0's.
  - Set the preceding bit to 1.
- Runtime: $\Theta(b)$, where b is the number of bits flipped.

An Amortized Analysis

- **Claim:** Starting at zero, the amortized cost of adding one to the total is O(1).
- **Idea:** Use as a potential function the number of 1's in the number.

$\Phi = 2$ 0 0 0 1 1

An Amortized Analysis

- **Claim:** Starting at zero, the amortized cost of adding one to the total is O(1).
- **Idea:** Use as a potential function the number of 1's in the number.

$\Phi = 1$ 0 0 1 0 0

Actual cost: 3
$\Delta\Phi$: -1

Amortized cost: **2**

Properties of Binomial Heaps

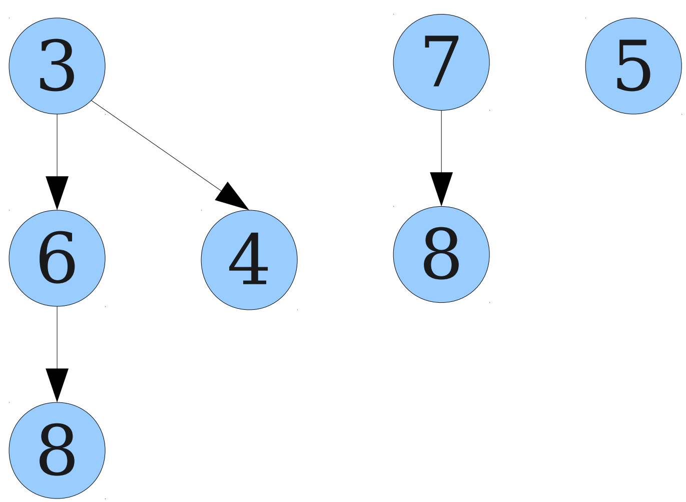
- Starting with an empty binomial heap, the amortized cost of each insertion into the heap is O(1), assuming there are no deletions.
- **Rationale:** Binomial heap operations are isomorphic to integer arithmetic.
- Since the amortized cost of incrementing a binary counter starting at zero is O(1), the amortized cost of enqueuing into an initially empty binomial heap is O(1).

Binomial vs Binary Heaps

- Interesting comparison:
  - The cost of inserting n elements into a binary heap, one after the other, is $\Theta(n \log n)$ in the worst-case.
  - If n is known in advance, a binary heap can be constructed out of n elements in time $\Theta(n)$.
  - The cost of inserting n elements into a binomial heap, one after the other, is $\Theta(n)$, even if n is not known in advance!

A Catch

- This amortized time bound does not hold if ***enqueue*** and ***extract-min*** are intermixed.
- **Intuition:** Can force expensive insertions to happen repeatedly.



Question: Can we make insertions amortized O(1), regardless of whether we do deletions?

Where's the Cost?

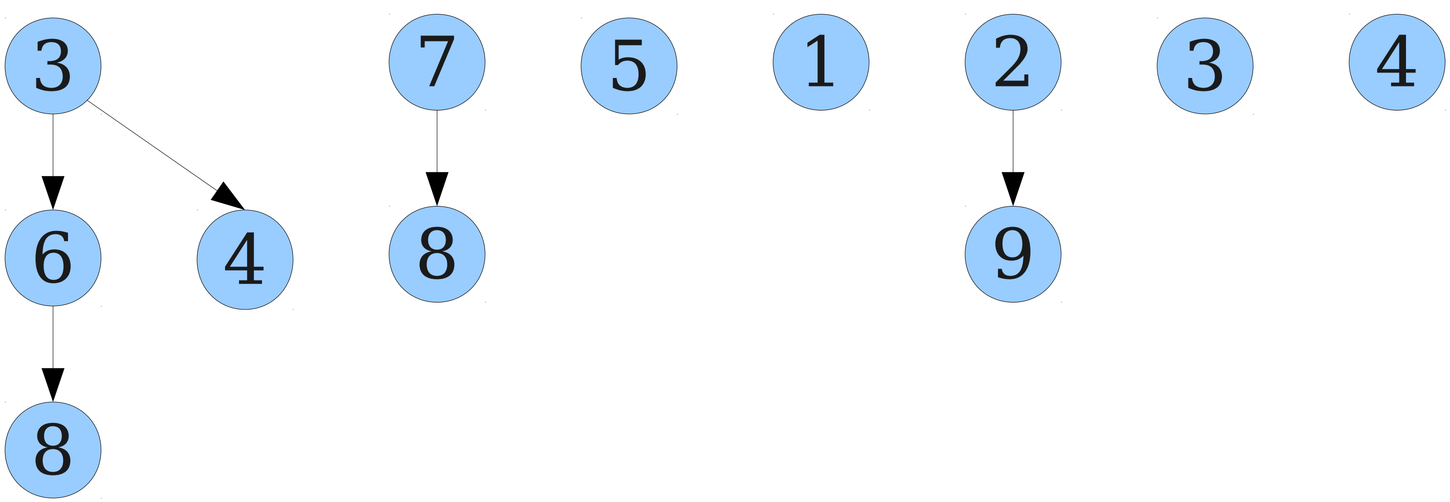
- Why does ***enqueue*** take time O(log *n*)?
- **Answer**: May have to combine together O(log *n*) different binomial trees together into a single tree.
- **New Question**: What happens if we don't combine trees together?
- That is, what if we just add a new singleton tree to the list?

Lazy Melding

- More generally, consider the following lazy melding approach:

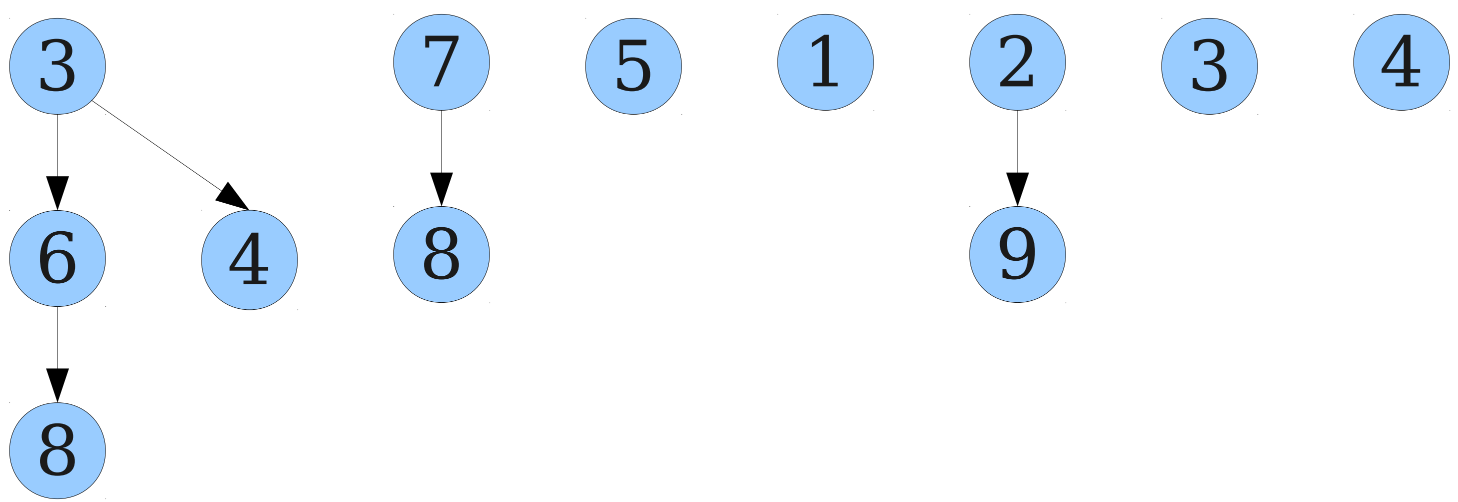
To meld together two binomial heaps, just combine the two sets of trees together.

- If we assume the trees are stored in doubly-linked lists, this can be done in time O(1).



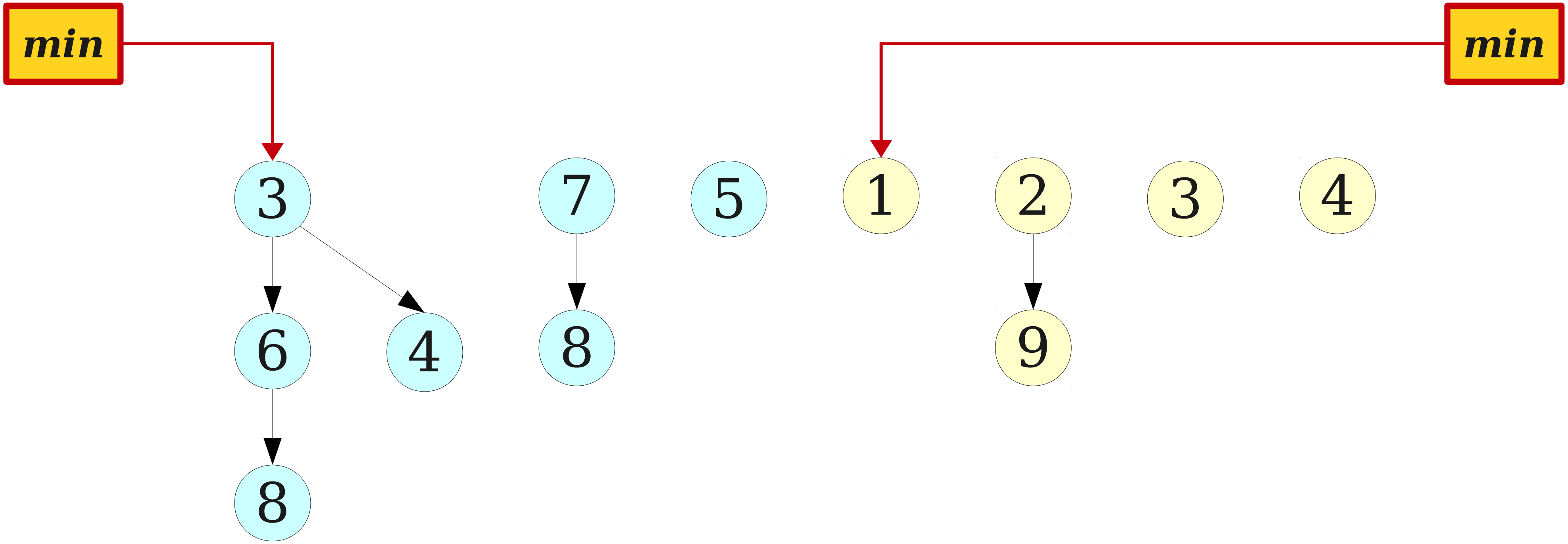
The Catch: Part One

- When we use eager melding, the number of trees is O(log n).
- Therefore, ***find-min*** runs in time O(log n).
- **Problem:** ***find-min*** no longer runs in time O(log n) because there can be Θ(n) trees.



A Solution

- Have the binomial heap store a pointer to the minimum element.
- Can be updated in time O(1) after doing a meld by comparing the minima of the two heaps.



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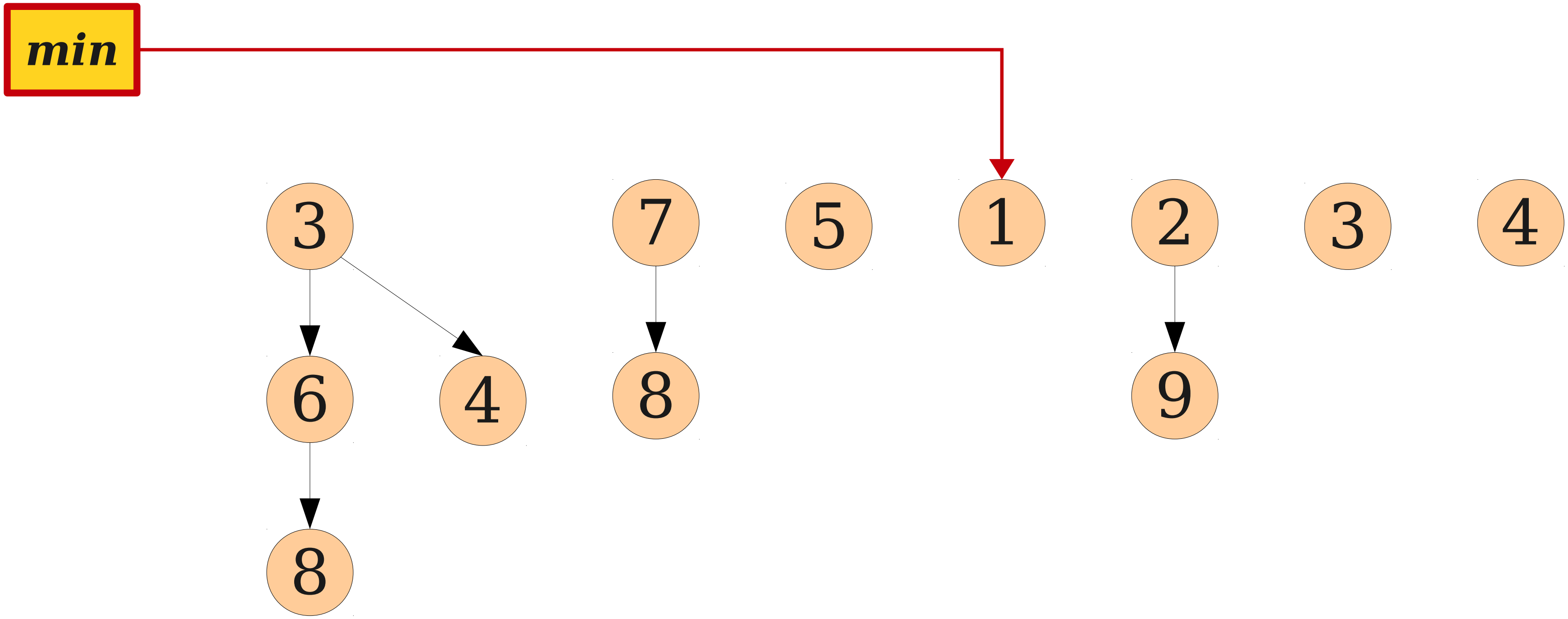
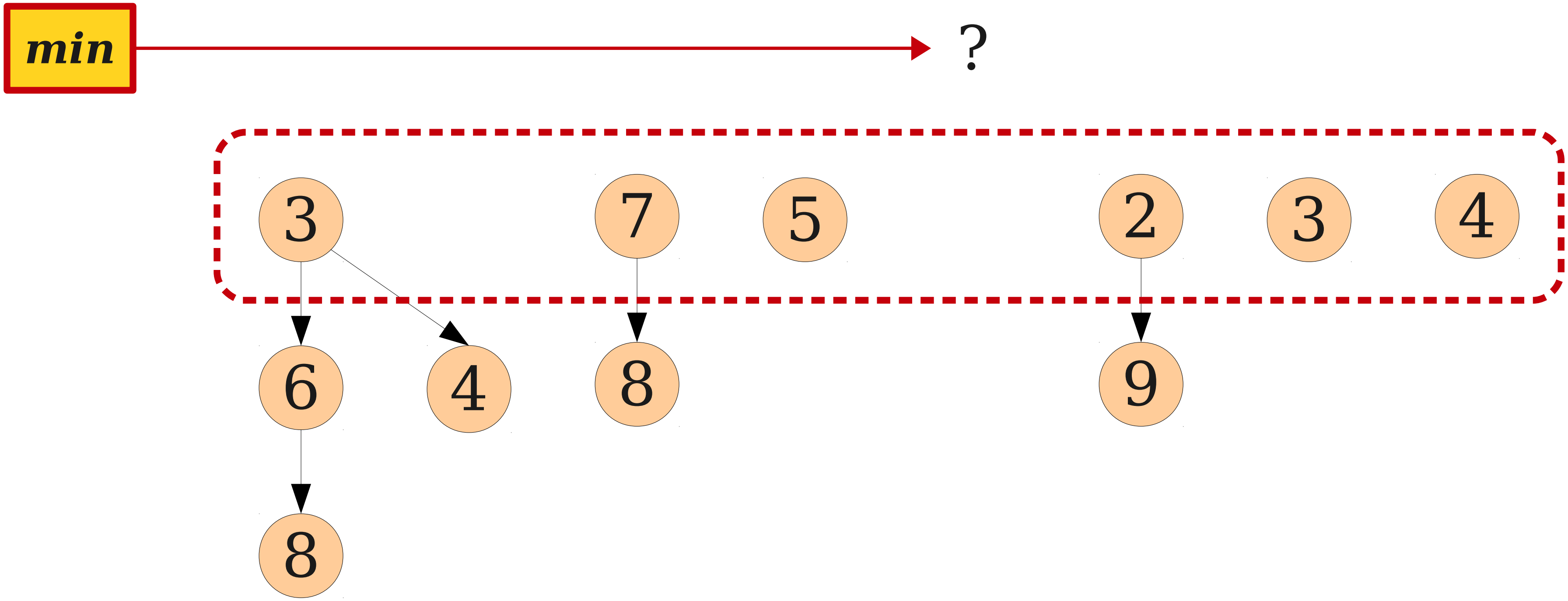


The Catch: Part Two

- Even with a pointer to the minimum, deletions might now run in time $\Theta(n)$.
- **Rationale:** Need to update the pointer to the minimum.

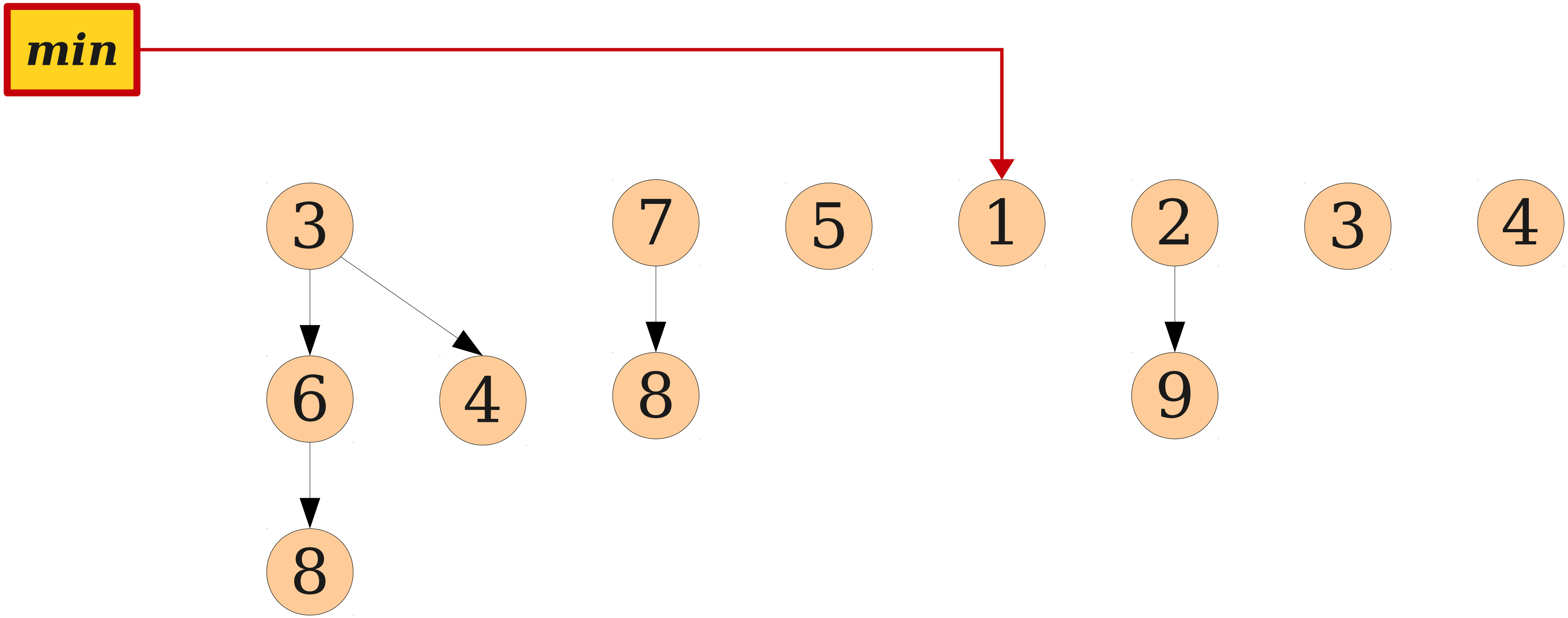


Resolving the Issue

- **Idea:** When doing an ***extract-min***, coalesce all of the trees so that there's at most one tree of each order.
- Intuitively:
  - The number of trees in a heap grows slowly (only during an insert or meld).
  - The number of trees in a heap drops rapidly after coalescing (down to O(log n)).
  - Can backcharge the work done during an ***extract-min*** to ***enqueue*** or ***meld***.

Coalescing Trees

- Our eager melding algorithm assumes that
  - there is either zero or one tree of each order, and that
  - the trees are stored in ascending order.
- **Challenge:** When coalescing trees in this case, neither of these properties necessarily hold.



Wonky Arithmetic

- Compute the number of bits necessary to hold the sum.
  - Only O(log n) bits are needed.
- Create an array of that size, initially empty.
- For each packet:
  - If there is no packet of that size, place the packet in the array at that spot.
  - If there is a packet of that size:
    - Fuse the two packets together.
    - Recursively add the new packet back into the array.

Now With Trees!

- Compute the number of *trees* necessary to hold the *nodes*.
  - Only O(log n) *trees* are needed.
- Create an array of that size, initially empty.
- For each *tree*:
  - If there is no *tree* of that size, place the *tree* in the array at that spot.
  - If there is a *tree* of that size:
    - Fuse the two *trees* together.
    - Recursively add the new *tree* back into the array.

Analyzing Coalesce

- Suppose there are T trees.
- We spend $\Theta(T)$ work iterating across the main list of trees twice:
  - Pass one: Count up number of nodes (if each tree stores its order, this takes time $\Theta(T)$).
  - Pass two: Place each node into the array.
- Each merge takes time O(1).
- The number of merges is $O(T)$.
- Total work done: $\boldsymbol{\Theta(T)}$.
- In the worst case, this is $O(n)$.

The Story So Far

- A binomial heap with lazy melding has these worst-case time bounds:
  - ***enqueue***: O(1)
  - ***meld***: O(1)
  - ***find-min***: O(1)
  - ***extract-min***: O(n).
- These are *worst-case* time bounds. What about an *amortized* time bounds?

An Observation

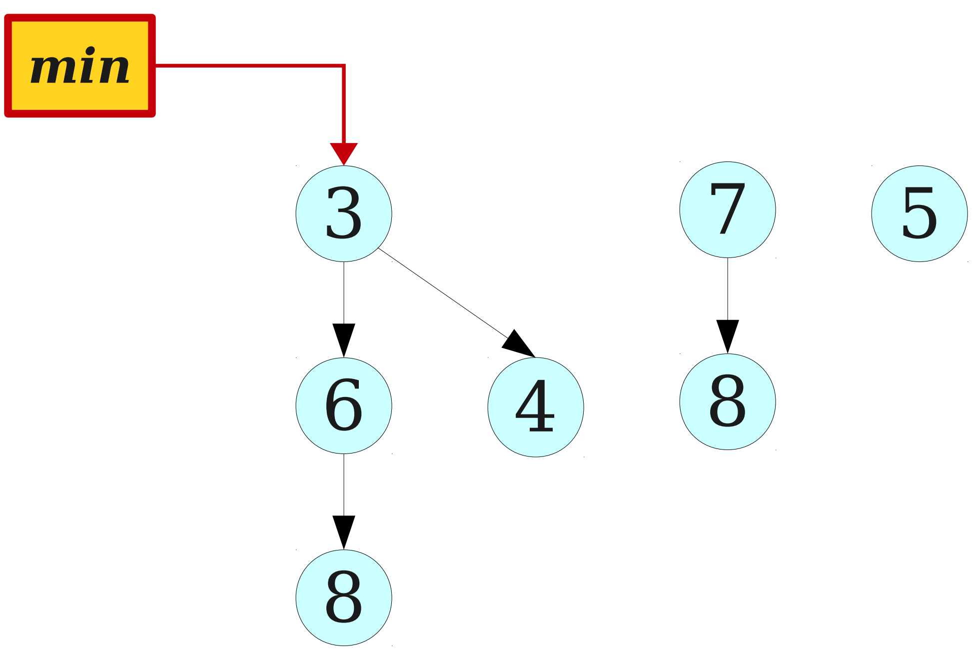
- The expensive step here is ***extract-min***, which runs in time proportional to the number of trees.
- Each tree can be traced back to one of three sources:
  - An ***enqueue***.
  - A ***meld*** with another heap.
  - A tree exposed by an ***extract-min***.
- Let's use an amortized analysis to shift the blame for the ***extract-min*** performance to other operations.

The Potential Method

- We will use the potential method in this analysis.
- When analyzing insertions with eager merges, we set $\Phi(D)$ to be the number of trees in D.
- Let's see what happens if we use this Φ here.

Analyzing an Insertion

- To ***enqueue*** a key, we add a new binomial tree to the forest and possibly update the *min* pointer.

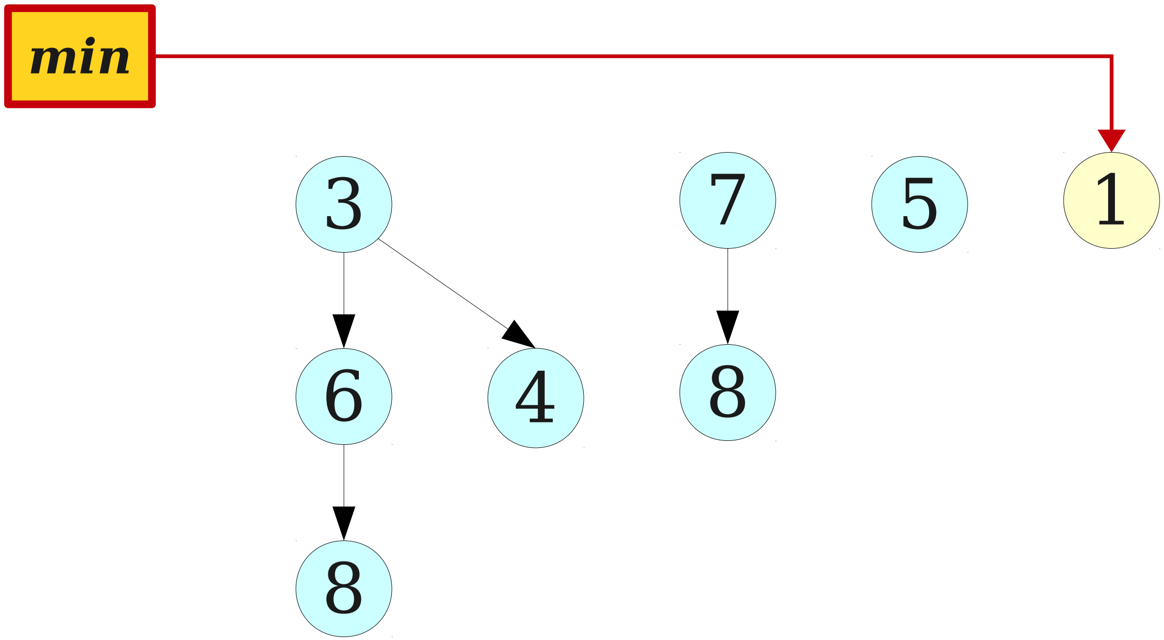


Analyzing an Insertion

- To ***enqueue*** a key, we add a new binomial tree to the forest and possibly update the *min* pointer.

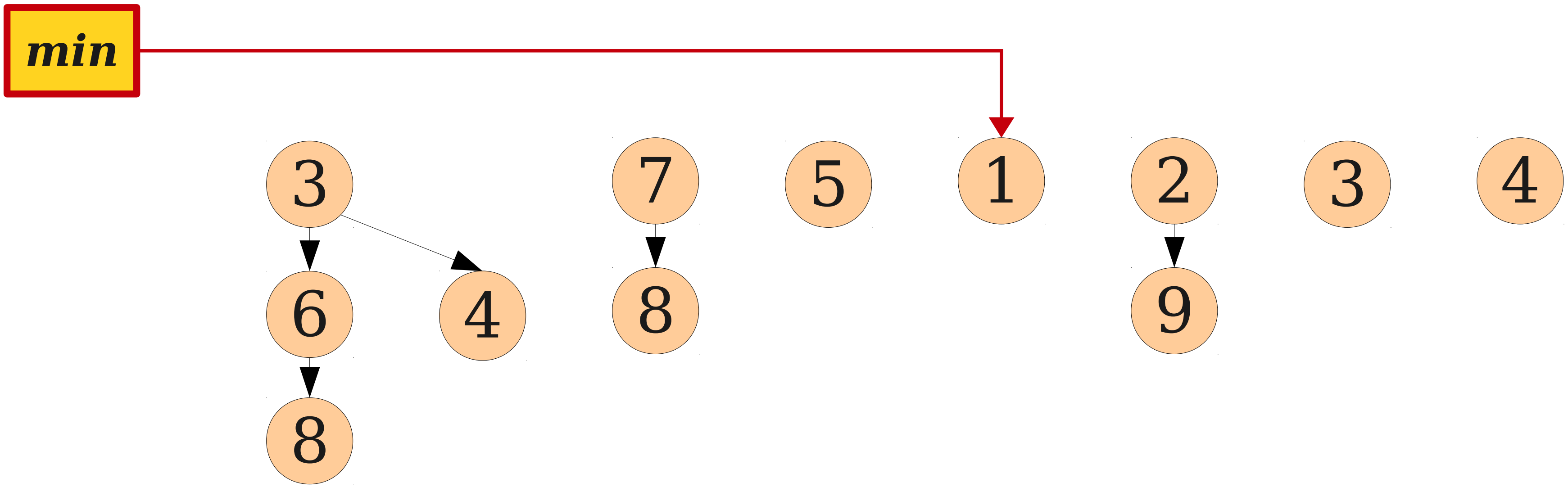
  Actual time: O(1). ΔΦ: +1

  Amortized time: **O(1)**.



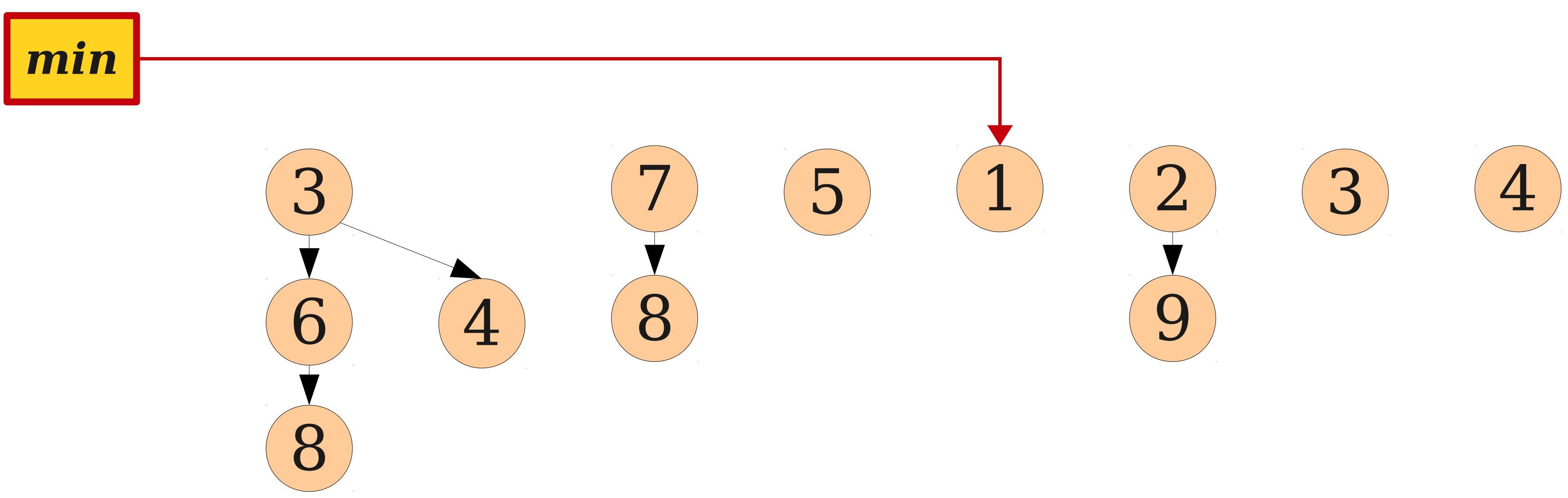
Analyzing a Meld

- Suppose that we ***meld*** two lazy binomial heaps B_1 and B_2. Actual cost: O(1).
- Let Φ_{B_1} and Φ_{B_2} be the initial potentials of B_1 and B_2.
- The new heap B has potential $\Phi_{B_1} + \Phi_{B_2}$ and B_1 and B_2 have potential 0.
- $\Delta\Phi$ is zero.
- Amortized cost: **O(1)**.



Analyzing a Find-Min

- Each ***find-min*** does O(1) work and does not add or remove trees.
- Amortized cost: **O(1)**.



Analyzing Extract-Min

- Initially, we expose the children of the minimum element. This takes time $O(\log n)$.
- Suppose that at this point there are T trees. As we saw earlier, the runtime for the coalesce is $\Theta(T)$.
- When we're done merging, there will be $O(\log n)$ trees remaining, so $\Delta\Phi = -T + O(\log n)$.
- Amortized cost is

$$O(\log n) + \Theta(T) + O(1) \cdot (-T + O(\log n))$$

$$= O(\log n) + \Theta(T) - O(1) \cdot T + O(1) \cdot O(\log n)$$

$$= \mathbf{O(\log \mathit{n})}.$$

The Overall Analysis

- The *amortized* costs of the operations on a lazy binomial heap are as follows:
  - ***enqueue***: O(1)
  - ***meld***: O(1)
  - ***find-min***: O(1)
  - ***extract-min***: O(log n)
- Any series of e ***enqueues*** mixed with d ***extract-min***s will take time $O(e + d \log e)$.

Why This Matters

- Lazy binomial heaps are a powerful building block used in many other data structures.
- We'll see one of them, the *Fibonacci heap*, when we come back on Wednesday.
- Assuming the TAs think it's reasonable, you'll see another (supporting ***add-to-all***) on the problem set.

Next Time

- **The Need for decrease-key**
  - A powerful and versatile operation on priority queues.
- **Fibonacci Heaps**
  - A variation on lazy binomial heaps with efficient decrease-key.
- **Implementing Fibonacci Heaps**
  - … is harder than it looks!