9901 Geometric Computing	Lecture 1 - 01/01/2010

Euclidean Geometry Essentials

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In this lecture we review some basic facts from computational geometry (point geometry, in particular) and start considering basic constructive methods for curves and surfaces. We discuss curves and surfaces in more detail in future lectures.

1 Point

In computational geometry, geometric objects are usually sets of points in Euclidean space. By associating a coordinate system to Euclidean space we get a vector space; consequently, each point can be represented as a vector of cartesian coordinates of a given dimension.

Geometric objects do not necessarily consist of finite sets of points, but must be finitely described somehow. For example, a straight line can be defined by two points.

Algebraic Representation

Let $\mathbb{R}^n$ be the *n-dimensional Euclidean space*, i.e., the space of the n-tuples $(x_1, \ldots, x_n)$ of real numbers x_i, $i = 1, \ldots, n$, called cartesian coordinates, with metric $(\sum_{i=1}^n x_i^2)^{1/2}$. The algebraic representation of a point $P \in \mathbb{R}^n$ is just a n-tuple $(x_1, \ldots, x_n)$ of real numbers x_i, $i = 1, \ldots, n$. Thus, a *point* represents a location in space.

Vector Representation

In an equivalent manner, a point $P = (x_1, \ldots, x_n)$ can be interpreted as a n-component vector starting at the origin $(0, \ldots, 0)$ of $\mathbb{R}^n$ and ending at the point $(x_1, \ldots, x_n)$. Such a vector is also known as an oriented straight line.

2 Point Operations

The well-known arithmetic operations on real numbers of $\mathbb{R}$ can be easily extended to points in $\mathbb{R}^n$.

Addition

The addittion of two points $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of $\mathbb{R}^n$ is defined as follows:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \tag{1}$$

Subtraction

This is a particular of addition:

$$(x_1, \ldots, x_n) - (y_1, \ldots, y_n) = (x_1 - y_1, \ldots, x_n - y_n) \tag{2}$$

Product by a Scalar

The product of a scalar α by a point is another point:

$$\alpha\,(x_1, \dots, x_n) = (\alpha\, x_1, \dots, \alpha\, x_n) \tag{3}$$

As a consequence of the addition and product by a scalar for points, we have the following properties:

1. $(\alpha\,\beta)(x, y, z) = \alpha[\beta(x, y, z)]$ (associativity)
2. $(\alpha\, + \beta)(x, y, z) = \alpha(x, y, z) + \beta(x, y, z)$ (distributivity)
3. $\alpha[(x, y, z) + (x', y', z')] = \alpha(x, y, z) + \alpha(x', y', z')$ (distributivity)
4. $\alpha(0, 0, 0) = (0, 0, 0)$ (absorvent element)
5. $0(x, y, z) = (0, 0, 0)$ (absorvent element)
6. $1(x, y, z) = (x, y, z)$ (identity element)

Multiplication

The product of two point is defined as follows:

$$(x_1, \dots, x_n).(y_1, \dots, y_n) = (x_1.y_1, \dots, x_n.y_n) \tag{4}$$

Exercise L1.1 Let $P = (x, y, z)$ and $Q = (x', y', z')$ be two points in $\mathbb{R}^3$. The Euclidean distance from P to Q is given by $d(P, Q) = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$. Show that $d(P, Q) = \sqrt{(P - Q)^2}$.

Resolution: By applying the previous properties for points, we have:

$$\begin{aligned}(P - Q)^2 &= [(x, y, z) - (x', y', z')]^2 \\ &= (x - x', y - y', z - z')^2 \\ &= (x - x', y - y', z - z').(x - x', y - y', z - z') \\ &= (x - x')^2 + (y - y')^2 + (z - z')^2\end{aligned}$$

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3 Midpoint

The midpoint or center C of two points P and Q satisfies the the following equality:

$$d(P, C) = d(Q, C) = \frac{1}{2} d(P, Q)$$

Theorem 1. *For every pair $P, Q \in \mathbb{R}^3$, there exists a single center $C = \frac{1}{2}(P + Q)$.*

Proof: We know that $d(P, Q) = |P - Q|$, so by hypothesis we have:

$$d(P, C) = |P - \frac{1}{2}(P + Q)|$$

that is,

$$\begin{aligned}
d(P,C) &= \frac{1}{2}|P-Q| \\
&= \frac{1}{2}|Q-P| \\
&= |Q-\frac{1}{2}(P+Q)| \\
&= |Q-C| \\
&= d(Q,C)
\end{aligned}$$

from where follows that $C=\frac{1}{2}(P+Q)$ is the only center of the pair P,Q.

4 Barycentric Combinations

The formulation of barycentric coordinates was originated in physics, simply because barycenter means center of gravity.

The *barycentric combination* is considered the most fundamental operation of points. By definition, the barycentric combination is a weighted sum of points, whose weights sum up to 1, that is,

$$P=\sum_{i=0}^{k}\alpha_i\,P_i \tag{5}$$

where $P_i\in\mathbb{R}^n$ and $\alpha_0+\alpha_1+\ldots+\alpha_d=1$.

Examples: Two examples are:

1. Center of a pair of points.
2. Centroid C of a triangle defined by three points, P, Q, and R:

$$C=\frac{1}{3}P+\frac{1}{3}Q+\frac{1}{3}R$$

5 Convex Combinations, convex set, and convex hull

A *convex combination* is a particular case of barycentric combination such that all $\alpha_i\geq 0$.

Examples: Accordingly, we have:

1. The convex combination of two points P, Q is a point of the straight line segment (P,Q).
2. The convex combination of three points P, Q, R is a point of the triangle (P,Q,R).

This leads us to the following:

Definition 2. *The* convex hull *of a set of points is defined as the set of all convex combinations of those points.*

For each collection of coefficients $\alpha_i\geq 0$, we obtain a point of the convex hull. For example, a triangle is the convex hull of three points, so that each one of its points can be obtained from a different collection of coefficients $\alpha_i\geq 0$. More precisely, an interior point of a triangle satisfy the inequality $\alpha_i>0$ for the corresponding three coefficients, whereas a frontier point of the triangle satisfy $\alpha_i=0$ for only one coefficient.

In more intuitive terms, the convex hull of a set of points is the smallest convex set that contains those points. Recall that, a *convex set* is defined as ....

6 Flat Euclidean Primitives

7 Affine Transformations

The term of *affine map* is due to Euler. Well-known affine transformations in computer graphics are, for example, identity, translation, rotation, scaling, and shearing. In computer graphics, we define affine transformations in terms of linear transformations.

Here, we define affine transformations in terms of barycentric combinations as follows:

Definition 3. *A map f from $\mathbb{R}^n$ to $\mathbb{R}^n$ is an* affine transformation *if f let the barycentric combinations invariant.*

So, if $P \in \mathbb{R}^n$ is the result of a barycentric combination of N points P_i in $\mathbb{R}^n$, that is,

$$P = \sum_{i=1}^{N} \alpha_i \, P_i$$

and $f : \mathbb{R}^n \to \mathbb{R}^n$ is an affine transformation, then

$$f(P) = \sum_{i=1}^{N} \alpha_i \, f(P_i),$$

with $f(P), f(P_i) \in \mathbb{R}^3$.

Examples: Two examples are:

1. The midpoint of a straight line segment is transformed into the midpoint of its image.
2. The centroid of a set of points is transformed into the centroid of its image.

Matrix Form

An affine transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ between two affine spaces consists of a linear transformation followed by a translation as follows:

$$f(P) = \mathbf{M}\, P + Q \tag{6}$$

where $\mathbf{M}$ is a $n \times n$ matrix and P, Q are points in $\mathbb{R}^n$. (Note that, usually, affine transformations are formulated in terms of vectors, instead of points.)

Exercise L1.2 Let us prove that an affine transformation given by formula (6) preserve the barycentric combinations.

Resolution: By replacing $P = \sum_{i=1}^{N} \alpha_i \, P_i$ in (6) we get:

$$\begin{aligned} f(\sum_{i=1}^{N} \alpha_i \, P_i) &= \mathbf{M} \, (\sum_{i=1}^{N} \alpha_i \, P_i) + Q \\ &= \sum_{i=1}^{N} \alpha_i \, \mathbf{M} \, P_i + 1 \, . \, Q \\ &= \sum_{i=1}^{N} \alpha_i \, \mathbf{M} \, P_i + \sum_{i=1}^{N} \alpha_i \, . \, Q \\ &= \sum_{i=1}^{N} \alpha_i \, (\mathbf{M} \, P_i + Q) \\ &= \sum_{i=1}^{N} \alpha_i \, f(P_i) \end{aligned}$$

Examples: Let us see some affine transformations in $\mathbb{R}^3$:

1. *Identity*: $\mathbf{M} = \mathbf{I}$ and $Q = (0, 0, 0)$.
2. *Translation*: $\mathbf{M} = \mathbf{I}$ and $Q = (x, y, z)$.
3. *Scaling*: $\mathbf{M}$ is a diagonal matrix and $Q = (0, 0, 0)$.
4. *Rotation about z-axis*: $\mathbf{M} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $Q = (0, 0, 0)$.
5. *Shearing*: $\mathbf{M} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ and $Q = (0, 0, 0)$.

Euclidean transformations are particular affine transformations, and are known rigid transformations. Translation and rotation are Euclidean transformations. The matrix $\mathbf{M}$ of an Euclidean transformation is orthonormal, that is, $\mathbf{M}^T \, \mathbf{M} = \mathbf{I}$. Euclidean transformations preserve distances and angles.

8 Classification of Geometries

In this lecture, we have approached two geometries with reference to maps applied to a set of points: Euclidean geometry and affine geometry.

Euclidean geometry can be defined as a pair of two sets: a set of points and the set of Euclidean transformations. Its main invariant is the distance between points.

Affine geometry is more general because includes more than Euclidean transformations; for example, it admits the scaling transformation. Therefore, it does not preserve the distance between points

A still more general geometry is the *projective geometry*, which is the mathematical model behind any 3D graphics system. Its main invariant is ...

In short, we have:

$$\text{Euclidean geometry} \subset \text{affine geometry} \subset \text{projective geometry}$$

9 Final Remarks

References

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[2] H. Rahul, N. Kushman, D. Katabi, C. Sodini, and F. Edalat, "Learning to Share: Narrowband-Friendly Wideband Wireless Networks", *ACM SIGCOMM Computer Communication Review*, vol. 38, no. 4, pp. 147–158, 2008.